AMATH 250 with Shahla Aliakbari*

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*There is no assignment in this course, the grade is consisted of 10 quizzes (%20), one midterm (30%), and a final exam (%50)

Contents

1 First Order Differential Equations

Lecture 1 - Thursday, January 08

Definition 1.1. **[Differential Equation]**

A **differential equation** is an equation for an unknown function involving one or more derivatives for an unknown function.

Example 1.1. Here is an example:

$$f(x, y(x), y'(x)) = 0$$

where x is an independent variable, $y(x)$ is a dependent variable, and $y'(x)$ is the first derivative of the dependent variable.

Example 1.2. Here is a more concrete example:

$$y'y + \sin(y) = x^2, \qquad y = y(x),\ y' = \frac{dy}{dx}$$

is a differential equation.

We have three types of differential equations[1]:

1. ODE;
2. PDE;
3. SDE.

Definition 1.2. **[ODE (Ordinary Differential Equation)]**

A differential equationf with one dependent variable and one independent variable is called a **ordinary differential equation**.

Example 1.3.

$$\frac{dy}{dx} + 2y\tan(x) = 0, \qquad y = y(x)$$

is an ordinary differential equation.

Definition 1.3. **[PDE (Partial Differential Equation)]**

A differential equationf with one dependent variable and two or more independent variables is called a **partial differential equation**.

[1]In this course, we will be learning only two of them

Example 1.4.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u = u(t, x)$$

is a partial differential equation.

Example 1.5.

$$\frac{\partial y}{\partial t} = 4 \cdot \frac{\partial^2 y}{\partial x^2} + t\tan(x) = 0, \qquad y = y(t, x)$$

is a partial differential equation.

Definition 1.4. **[SDE (System of Differential Equations)]**

A differential equation is a **system of differential equations** if there are two or more dependent variables and only one independent variable.

Example 1.6.

$$\begin{cases} \frac{dy}{dt} = 2y - 3x & \qquad y = y(t) \\ \frac{dx}{dt} = y + 2x & \qquad x = x(t) \end{cases}$$

is a system of differential equations.

1.1 Linearity and Order of Differential Equations

Definition 1.5. **[Order]**

The **order** of a differential equation is the order of the highest derivative in the equation.

Example 1.7. Some examples:

- $y'' + y^4 = \cos(x)$ is an ODE of order 2.
- $y'' + y^{(4)} = \cos(x)$ is an ODE of order 4.
- $(y')^2 + \sin(y) = e^x$ is an ODE of order 1.
- $u_{tt} - 2u_{xx} = 0$ is an ODE of order 2.

Definition 1.6. **[Linearity]**

A differential equation is **linear** if the dependent variable and its derivatives appear linearly in the equation. Otherwise, the differential equation is non-linear.

Example 1.8. Some examples:

- $\sin(x)y'' + x^2y = e^x, \qquad y = y(x) \quad$ is a 2nd-order linear ODE.
- $yy' + x = 1 \quad$ is a 1st-order non-linear ODE.

Exercise 1.1. Classify the following DEs:

1. $x^2 \tan(x)y'' + 4x(y')^3 = 0$;
2. $xy' + 2|y| + \tan(x) = 0$;
3. $y'' + 2y' + \cos(t)y = t^2 + 6t + 9$;
4. $y''y''' + x^5y' = \sin(x)$;
5. $(y - x)dx + 2xdy = 0$.

Answer.

1. 2nd-order non-linear ODE;
2. 1st-order non-linear ODE;
3. 2nd-order linear ODE;
4. 3rd-order non-linear ODE;
5. If y is a dependent variable over x, this is a 1st-order linear ODE. Otherwise, this is a 1st-order non-linear ODE.

$\square$

Exercise 1.2. Classify the DEs:

1. $e^xy' + 3y = x^2y$;
2. $Au_{xx} + 2Bu_{xy} = 0$.

Answer. The first one is first-order linear ODE, while the second one is second-order linear PDE. $\square$

1.2 Solution of Differential Equations

Definition 1.7. **[Solution]**

A function f is a **solution** for an ODE if it satisfies the equation.

Note 1.1. Consider

$$f(x, y, y', y'', \ldots, y^{(n)}) = 0, \qquad y = y(x)$$

A function $u(x)$ defined on an interval such as $I \subseteq R$ is said to be a solution to the DE if $u(x)$ is n-time differentiable on I and satisfies the DE. i.e.,

$$f(x, u(x), u'(x), u''(x), \ldots, u^{(n)}(x)) = 0, \qquad \forall x \in I$$

The interval I is called the **interval of the solution**.

Example 1.9. Verify that the given functions are a solution to the corresponding DE:

1. $y' = x\sqrt{y}$, $y = y(x)$, $y(x) = 1/16 \cdot x^4$. We have

$$LHS = \frac{1}{4} \cdot x^3 = RHS$$

   and the interval of the solution is $\mathbb{R}$.

2. $y' = -(xy)/\ln(y)$, $y > 0$, $y(x) = \exp(\sqrt{4 - x^2})$. We have

$$LHS = \left((4 - x^2)^{1/2}\right)' \cdot e^{\sqrt{4-x^2}} = -\frac{x}{\sqrt{4 - x^2}} \cdot e^{\sqrt{4-x^2}}$$
$$RHS = -\frac{x \cdot e^{\sqrt{4-x^2}}}{\ln e^{\sqrt{4-x^2}}} = \frac{-xe^{\sqrt{4-x^2}}}{\sqrt{4 - x^2}}$$

   and the interval of the solution is $(-2, 2)$.

3. $y'' - y(x) = 0$, $y(x) = e^x$. This is easy to verify, and the interval of the solution is $\mathbb{R}$. We can also verify that $y = e^{-x}$ is another solution to the DE.

Note 1.2. The interval of the solution needs to satisfy both the solution and the differential equation.

Note 1.3. There can be multiple solutions to a single differential equation.

1.2.1 The General Solution to the Differential Equations

Theorem 1.1.

Informally speaking, a linear n-ordered differential equation "has n solutions" (generalized as a "single" function, see below).

The general solution to the linear n-ordered DEs is written in the form

$$y(x) = C_1 \cdot y_1(x) + C_2 \cdot y_2(x) + \cdots + C_n \cdot y_n(x)$$

where C_i are arbitrary constants and y_i are solutions.

Definition 1.8. **[General Solution]**

A **general solution** of an ODE is a family of functions which represent all (or nearly all) possible solutions. Occasionally, special solutions exist outside of the family.

1.3 Initial Value Problems and Boundary Value Problems

Lecture 2 - Friday, January 09

Definition 1.9. **[Initial Value Problem]**

Initial value problem: solving differential equations subject to given initial conditions.

Definition 1.10. **[Initial Condition]**

Initial conditions are values of function and its derivatives at one point of independent variable x_0.

Note 1.4. We can use initial conditions to find the aribitrary constants in the general solution.

Definition 1.11. **[Boundary Condition]**

Boundary conditions are the values of the unknown function and/or its derivatives at different values of independent variable.

Definition 1.12. **[Boundary Value Problem]**

Solving a differential equation subject to boundary conditions is called **boundary value problem**.

1.4 Solving First Order Differential Equations

Note 1.5. We refer to ODES as DEs (differential equations).

Note 1.6. A differential equation may not have any solution. Here is an example:

$$|y'| + |y| + 1 = 0$$

The general form of a first order DE is

$$f(x, y, y') = 0, \qquad y = y(x), \qquad y(x_0) = y_0$$

The standard form is

$$\frac{dy}{dx} + p(x)y(x) = g(x)$$

where the coefficient for the term $\frac{dy}{dx}$ is 1.

1.4.1 Integrating Factor: For First-order Linear Homogenous DEs

We are looking for an unknown function $\mu(x)$ such that if we multiply both sides of the DE by it, we can write the left-hand side as

$$\frac{d}{dx}\big(\mu(x)y(x)\big).$$

1. Multiply both sides (standard form) by $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y(x) = \mu(x)g(x).$$

> **Comment 1.1.** Recall that we want
>
> $$\frac{d}{dx}\big(\mu(x)y(x)\big) = \mu(x)g(x).$$

2. Use the product rule to expand the left-hand side. Recall that product rule is:

$$\frac{d\mu}{dx}\,y(x) + \mu(x)\frac{dy}{dx} = \mu(x)g(x).$$

3. Comparing with step 1 and 3, we obtain:

$$\begin{aligned}
\cancel{\mu(x)\frac{dy}{dx}} + \mu(x)p(x)y(x) &= \frac{d\mu}{dx}y(x) + \cancel{\mu(x)\frac{dy}{dx}} \\
\mu(x)\cancel{y(x)}p(x) &= \frac{d\mu}{dx}\cancel{y(x)} \\
\frac{d\mu}{u} &= p(x)dx
\end{aligned}$$

4. Integrate both sides:

$$\begin{aligned}
\int \frac{1}{\mu}\,d\mu &= \int p(x)\,dx + C \\
\ln|\mu| &= \int p(x)\,dx + C \\
|\mu| &= e^C\,e^{\int p(x)\,dx} \\
\mu(x) &= C_1\,e^{\int p(x)\,dx}.
\end{aligned}$$

We may take $C_1 = 1$ since it cancels when multiplying both sides by μ.

5. We have found $\mu(x)$. Now we solve:

$$\begin{aligned}
\frac{d}{dx}\big(\mu(x)y(x)\big) &= \mu(x)g(x) \\
\mu(x)y(x) &= \int \mu(x)g(x)\,dx + C \\
y(x) &= \frac{1}{\mu(x)}\int \mu(x)g(x)\,dx + \frac{C}{\mu(x)}
\end{aligned}$$

Input : First-order linear ODE $y'(x) + p(x)y(x) = g(x)$ (optionally with $y(x_0) = y_0$)
Output: Solution $y(x)$

Write the DE in standard form $y'(x) + p(x)y(x) = g(x)$
Compute the integrating factor $\mu(x) \leftarrow \exp\bigl(\int p(x)\,dx\bigr)$
Multiply both sides by $\mu(x)$
Rewrite the left-hand side as $\dfrac{d}{dx}\bigl(\mu(x)y(x)\bigr)$
Integrate both sides
Solve for $y(x)$
if *an initial condition* $y(x_0) = y_0$ *is given* **then**
  Substitute into the general solution to determine the constant C
end

Algorithm 1: Integrating Factor Procedure (to show on assessments)

Example 1.10. Solve the DE: $\frac{dy}{dx} = \frac{1}{x}y + x^2e^x, \qquad x > 0.$

1. $$\frac{dy}{dx} - \frac{1}{x}y = x^2e^x.$$

2. $$\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}.$$

3. $$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = xe^x.$$

4. $$\frac{d}{dx}\left(\frac{1}{x}y\right) = xe^x.$$

5. $$\frac{1}{x}y = \int xe^x\,dx + C.$$

6. $$\int xe^x\,dx = xe^x - e^x.$$

7. $$\frac{1}{x}y = xe^x - e^x + C \qquad \Longrightarrow \qquad y(x) = x^2e^x - xe^x + Cx.$$

Exercise 1.3. (a) Solve the IVP

$$\frac{dy}{dx} + 2xy = 2e^{-x^2}, \qquad y(0) = 2.$$

(b) Evaluate $\lim_{x\to\infty} y(x)$, if it exists.

(c) Does $y(x)$ attain a maximum value for $x > 0$?

(d) Sketch the graph of $y(x)$.

1.4.2 Homogenous and Inhomogenous Differential Equations

Definition 1.13. **[Homogenous/ Inhomogenous DE]**

For a linear DE of the standard form

$$y'(x) + p(x)y(x) = g(x).$$

- If $g(x) = 0 \Rightarrow$ **homogeneous** DE $\Rightarrow$ find homogeneous solution $y_h(x)$.
- If $g(x) \neq 0 \Rightarrow$ **inhomogeneous** DE $\Rightarrow$ general solution is

  $$y(x) = y_h(x) + y_p(x),$$

  where $y_h(x)$ is the homogeneous solution and $y_p(x)$ is a particular solution.

A **particular solution** is a solution that satisfies the DE and does not have arbitrary constants.

Lecture 3 - Tuesday, January 13

So far, we have the following ways of solving linear first-order inhomogeneous differential equations, which is integrating factors as established above. For the homogeneous case, we will see how today.

1.4.3 Principle of Superposition

Theorem 1.2. Principle of Superposition

Consider the standard form of a linear first-order DE:

$$y'(x) + p(x)y(x) = g(x)$$

If $y_1(x)$ is a solution for the DE $y'(x) + p(x)y(x) = g_1(x)$ and $y_2(x)$ is a solution for the DE $y'(x) + p(x)y(x) = g_2(x)$. Then for any constants C_1 and C_2, $y(x) := C_1y_1(x) + C_2y_2(x)$ is a solution to the DE

$$y'(x) + p(x)y(x) = C_1g_1(x) + C_2g_2(x)$$

Proof. Starting by $y(x) = C_1y_1(x) + C_2y_2(x)$, substituting it back into the DE yields us

$$\begin{aligned} LHS &= (C_1y_1 + C_2y_2)' + p(x)(C_1y_1 + C_2y_2) \\ &= C_1y_1' + C_2y_2' + C_1p(x)y_1 + C_2p(x)y_2 \\ &= C_1(y_1' + p(x)y_1) + C_2(y_2' + p(x)y_2) \\ &= C_1g_1(x) + C_2g_2(x) = RHS \end{aligned}$$

Hence $y(x)$ is indeed a solution to the DE. □

1.4.4 The Method of Undetermined Coefficients: For Linear First-order DEs

This method[2] is for the DEs of the (standard) form:

$$y'(x) + py(x) = g(x)$$

the conditions are

- constant coefficient DEs;
- $g(x)$ is one of the special functions: polynomial, exponential, sine, or cosine.

Here is the full procedure:

1. Letting $g(x) = 0 \Rightarrow$ solve homogeneous equation $y' + py(x) = 0$. Write the solution as $y_h(x)$.
2. Trial particular solution $y_p(x)$ based on $g(x)$ (educated guess).
3. Substituting $y_p(x)$ into the DE, and find the unknown constants.
4. Write the general solution $y(x) = y_h(x) + y_p(x)$.

Example 1.11. Suppose we have the following DE:

$$y' + 3y = 4\cos(2t)$$

We first inspect, and find that we are able to use the undetermined coefficient method (section 1.4.4):

1. Standard form:
$$y' + 3y = \cos(2t)$$
2. Solving for $y' + 3y = 0$ we get $y_h(t) = \pm e^C e^{-3t} =: C_1 e^{-3t}$;
3. Now we trial $y_p(t)$. We know that $g(t) = \cos(2t)$, hence we guess $y_p(t) = A\cos(2t) + B\sin(2t)$.
4. Substituting $y_p(x)$ into the DE we get
$$(A\cos(2t) + B\sin(2t))' + 3(A\cos(2t) + B\sin(2t)) = 4\cos(2t)$$
Solving for A and B we obtain: $A = \frac{12}{13}$ and $B = \frac{8}{13}$. i.e.,
$$y_p(t) = \frac{12}{13}\cos(2t) + \frac{8}{13}\sin(2t)$$
5. Now we have the solution to the DE (as a result of the theorem 1.2):
$$y(t) = y_h(t) + y_p(t) = C_1 e^{-3t} + \left[\frac{12}{13}\cos(2t) + \frac{8}{13}\sin(2t)\right]$$

[2] Here is a pretty useful resource: link

Example 1.12. Solve the following DE:

$$y' + 2y = 3\exp(-2t)$$

Again, after inspection, we discover that we can use the undetermined coefficients method 1.4.4:

1. Solving for the homogeneous case, $y' + 2y = 0$, we obtain $y_h(t) = C_1 e^{-2t}$ similar as above.

2. Since $g(t) = 3e^{-2t}$, we would guess $y_p(t) = Ae^{-2t}$. However, we find that if so

$$\left(Ae^{-2t}\right)' + 2\left(Ae^{-2t}\right) = 0$$

   This is because Ae^{-2t} is just a special form of our $y_h(t)$. Here we introduce the trick that will help us:

   > Take $y_p(t) = Ate^{-2t}$. We multiply the "usual" trial with a factor of the independent variable.

   Do it again, we solve for A:

$$\left(Ate^{-2t}\right)' + 2\left(Ate^{-2t}\right) = 3e^{-2t}$$

   and obtain $A = 3$.

3. Therefore, the solution to the DE is

$$y(x) = C_1 e^{-2t} + 3te^{-2t}$$

Comment 1.2. If the input function $(g(x))$ is a linear combination of the functions that are suitable for the method of undetermined coefficients (or of product of these functions), then we use a combination of functions and their derivatives in our trial y_p with arbitrary constants that need to be determined.

Example 1.13. If $g(t) = 3t\sin(t) - 2e^{2t}$, we find three main components in this function:

- polynomial: t. For this, our trial is of the form $at + b$;
- sine: $\sin(t)$. For this, out trial is of the form $c\sin(t) + d\cos(t)$.
- exponental.

Therefore, our trial $y_p(t)$ will be of the form

$$y_p(t) = (At + B)\sin(t) + (Ct + D)\cos(t) + Ee^{2t}$$

Example 1.14. Solve the DE:
$$y' - y = e^t + t^2 + 2t\cos(t)$$

1. $y' - y = 0$ has solution $y_h(t) = C_1 e^t$;

2. Trial is $y_p(t) = Ate^t + \big(Bt^2 + Ct + D\big) + \Big[(Et + F)\cos(t) + (Gt + H)\sin(t)\Big]$. This function is quite big, so we can use the principle of superposition 1.2 to solve this DE:

$$\begin{array}{lllll} y' - y = e^t & \Rightarrow & y_{p_1}(t) = Ate^t & \Rightarrow & y_{p_1}(t) = te^t \\ y' - y = t^2 & \Rightarrow & y_{p_2}(t) = Bt^2 + Ct + D & \Rightarrow & y_{p_2}(t) = -t^2 - 2t - 2 \\ y' - t = 2t\cos(t) & \Rightarrow & y_{p_3}(t) = \text{(too long)} & \Rightarrow & y_{p_3}(t) = -t\cos(t) + (t+1)\sin(t) \end{array}$$

   and $y_p(t) = y_{p_1}(t) + y_{p_2}(t) + y_{p_3}(t)$.

3. Hence the general solution is

$$y(t) = C_1 e^t + te^t - t^2 - 2t - 2 - t\cos(t) + (t+1)\sin(t)$$

1.4.5 Separable Differential Equations

Definition 1.14. **[Separable]**

Separable differential equations are differential equations of the form

$$\frac{dy}{dx} = h(x)g(y)$$

Separable differential equations are nice because we can solve for y fairly easily:

$$\int \frac{dy}{g(y)} = \int h(x)dx + C$$

Example 1.15. $y' = e^{x+y}$ is separable.

1.5 Summary Linear First Order Differential Equations

Lecture 4 - Tuesday, January 20

Given a first order linear differential equation:

$$y'(x) + p(t)y(t) = g(t)$$

If the differential equation is homogeneous, we simply have

$$y_h(t) = c_1 e^{-\int p(t)dt}$$

If on the other hand it is inhomogeneous, we have two methods for different situation: integrating factor and undertermined coefficients. In the method of undertermined coefficients, the question is of the form

$$y' + py = g(t)$$

where p is a constant, and $g(t)$ is a special function. For picking the trial solution, consult the following table:

Input $g(t)$	**Trial** $y_p(t)$
$ae^{\alpha t}$ for $\alpha \neq -p$	$Ae^{\alpha t}$
$ae^{\alpha t}$ for $\alpha = -p$	$Ate^{\alpha t}$
$at^2 + bt + c$ $(p \neq 0)$	$At^2 + BT + C$
$at^2 + bt + c$ $(p = 0)$	$t(At^2 + BT + C)$
$a\sin(wt) + b\cos(wt)$	$A\sin(wt) + B\cos(wt)$
$at\sin(wt) + bt\cos(wt)$	$(At + B)\sin(wt) + (Ct + D)\cos(wt)$
$ae^{\alpha t}\sin(wt)$ (or $\cos(wt)$)	$e^{\alpha t}(A\sin(wt) + B\cos(wt))$

The last case is when the differential equation is separable, which is a easy case to deal with, example below:

Example 1.16. Solve the IVP: $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $y(0) = 2$.

Solution. Observe that this differential equation is separable:

$$y\,\mathrm{d}y = -x\,\mathrm{d}x \quad \Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C$$

Without the initial condition, we know that

$$y(x) = \pm\sqrt{x^2 + C}$$

for some constant C. Using the initial condition, we can determine that

$$y(x) = \sqrt{-x^2 + 4}$$

as desired. As a side note, we can find that the potential interval of the solution is $[-2, 2]$. However, since y appears in the denominator (on the RHS of the differential equation), we find that the actual interval of solution is $(-2, 2)$. $\square$

1.5.1 Application of Separable Differential Equations

Consider the activity of sky diving, where we have the following formula from Newton's Second law:

$$F_d - mg = ma$$

where $F_d = \beta v^2$. Rewrite the equation, we have the following IVP:

$$\beta v^2 - mg = m\frac{dv}{dt}, \quad v(0) = 0$$

Rearranging the equation we find that this is a separable differential equation:

$$\frac{dv}{(a^2 - v^2)} = -\frac{\beta}{m}dt$$

where $a = \frac{mg}{\beta}$.

Comment 1.3. A non-separable differential equation may be reduced to a separable differential equation by using appropriate change of variable:

1. $y' = f(ax + by)$ where a and b are constant. We can take $u = ax + by$ and thus

$$u' = \frac{du}{dx} = a + b\left(\frac{dy}{dx}\right)$$

so now the differential equation can be written as a separable differential equation:

$$\frac{du}{dx} = a + bf(u)$$

2. $y' = f(y/x)$. We can take $u = y/x$ and so $y = ux$, now

$$y' = u + xu'$$

now the differential equation becomes

$$xu' = f(u) - u \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}\left(f(u) - u\right)$$

which is indeed separable:

$$\frac{du}{f(u) - u} = \frac{dx}{x}$$

Example 1.17. Solve the differential equation:

$$\frac{dy}{dx} = (y - x)^2, \qquad y = y(x)$$

Solution. We first transform the problem to be separable, introduce new variabel $u \coloneqq y - x$, so we have

$u' = y' - 1$. Now

$$u' + 1 = u^2 \quad \Rightarrow \quad \frac{\mathrm{d}u}{u^2 - 1} = \mathrm{d}x \quad \Rightarrow \quad \left(\frac{-1/2}{u+1} + \frac{1/2}{u-1} \right) \mathrm{d}u = \mathrm{d}x$$

Solving for u we obtain

$$\int \left(\frac{-1/2}{u+1} + \frac{1/2}{u-1} \right) \mathrm{d}u = \int \mathrm{d}x + C \quad \Rightarrow \quad -\frac{1}{2} \ln|u+1| + \frac{1}{2} \ln|u+1| = x + C$$

Hence $\frac{u-1}{u+1} = C_1 e^{2x}$, and so $u = \frac{1+C_1e^{2x}}{1-C_1e^{2x}}$. This tells us that

$$y(x) = x + \frac{1 + C_1 e^{2x}}{1 - C_1 e^{2x}}$$

as desired. However, note that when we were solving the integral, we implicitly assumed that $u \neq \pm 1$. Hence we need to check for the case when $u = \pm 1$:

($u = 1$): In this case, we have $y = x + 1$, which is indeed covered in our general solution (take $C_1 = 0$);

($u = -1$): Now we have $y = x - 1$. This is a possible solution which is not included in our general solution. We need to report this **exceptional solution**.

In summary, the general solution to the differential equation is

$$y(x) = x + \frac{1 + C_1 e^{2x}}{1 - C_1 e^{2x}}, \qquad \text{and} \qquad y(x) = x - 1$$

as desired. □

Exercise 1.4. Show that $y(x) = x - 1$ is a solution to the differential equation in the previous example above.

Example 1.18. Solve the differential equation:

$$x\frac{dy}{dx} = y + \frac{x}{\sin\left(\frac{y}{x}\right)}$$

Solution. We observe that

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{\sin\left(\frac{y}{x}\right)}$$

Hence we can take $u = y/x$, $y' = u + xu'$:

$$\cancel{u} + xu' = \cancel{u} + \frac{1}{\sin u} \quad \Rightarrow \quad \sin u \, \mathrm{d}u = \frac{1}{x} \mathrm{d}x$$

Solving for u we obtain $u = \cos^{-1}(C_1 - \ln|x|)$, and hence

$$y(x) = x \cos^{-1}(C_1 - \ln|x|)$$

The only assumption we made is $x \neq 0$, which is automatic since it appears in the ln function. Hence this is our final solution to the differential equation. □

1.5.2 Qualitative Analysis of Differential Equations: Directional Fields

Lecture 5 - Tuesday, January 20

Definition 1.15. **[Directional Field]**

A directional field is a differential equation of the form

$$y' = f(x, y)$$

for *a differential equation* $y' = f(y)$, $y(x_0) = y_0$ **do**
  Find equilibrium solutions to $y' = 0 = f(y)$, call y_e;
  Find the sign of y' in the intervals $y > y_e$ and $e < y_e$;
  if $y' > 0$ **then**
    $y(x)$ is increasing;
  if $y' < 0$ **then**
    $y(x)$ is decreasing;
  From the differential equation, we obtain

$$\frac{dy'}{dx} = y'' = \frac{df(y)}{dx} = \frac{df(y)}{dy}\frac{dy}{dx}$$

  If $y'' = 0$, make a table for sign of y'' in the intervals;
  If $y'' > 0$, concave up;
  If $y'' > 0$, concave down;

Algorithm 2: Procedure for qualitative analysis of differential equations

Example 1.19. Consider the IVP $y'(x) = a(y - y_e)$, $y(0) = y_0$, $a \neq 0$, $y_e > 0$ are constants. Sketch the solution curves using directional field analysis.

Solution. We solve for $y' = 0 = a(y - y_2)$, which has solution $y = y_e$. Now we perform sign analysis: If $a > 0$, we have

	$y < y_e$	$y > y_e$
y'	negative	positive
y	decreasing	increasing

If $a < 0$,

	$y < y_e$	$y > y_e$
y'	positive	negative
y	increasing	decreasing

Compute y'':

$$y''(x) = ay' \cdot \frac{\mathrm{d}}{\mathrm{d}y}(y - y_e) = a^2(y - y_e)$$

We see that $y'' = 0$ when $y = y_e$. Pictures below depict the directional fields:

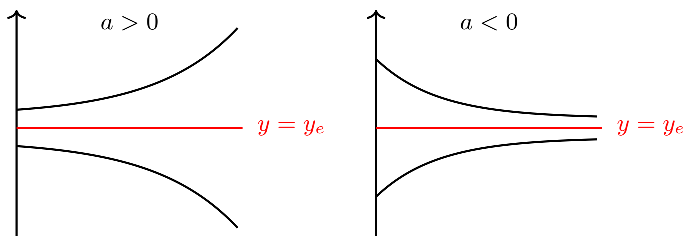


Figure. Solution-curve sketch for $y'(x) = a\,(y - y_e)$: unstable if $a > 0$, stable if $a < 0$.

$\square$

Exercise 1.5. Solve the differential equation $y'(x) = a(y - y_e)$, $y(0) = y_0$ and show that

$$y(x) = y_e + (y_0 - y_e)e^{ax}$$

and for $a < 0$, as $x \to \infty$, we have $y(x) \to y_e$.

Example 1.20. Consider the differential equation $p'(t) = 10p(6 - p)$ and have a qualitative sketch for the solution curves.

Solution. Solve for $p' = 10p(6 - p) = 0$, we have two equilibrium solutions: $p = 0$ and $p = 6$.

	$p < 0$	$0 < p < 6$	$6 < p$
p'	negative	positive	negative
p	decreasing	increasing	decreasing

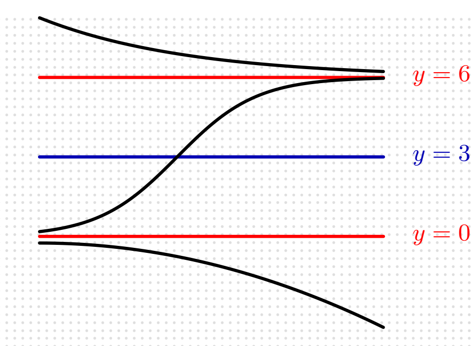
Compute p'':

$$p'' = p'\frac{df}{dp} = 10p(6 - p)\frac{d}{dp}(60p - 10p^2) = 10p(6 - p)(60 - 20p)$$

which has solutions $p = 0, 6, 3$:

	$p < 0$	$0 < p < 3$	$3 < p < 6$	$p > 6$
p''	negative	positive	negative	positive
p	down	up	down	up

We have the following picture:



$\square$

Exercise 1.6. Solve the differential equation above.

Note 1.7. For the differential equations $y'(x) = f(x, y)$, we follow the same procedure to sketch the solution curves.

Example 1.21. Use the directional field analysis to sketch the solution curves for $y' = y - t$, $y = y(t)$.

Solution. Solving for $y' = 0$, we obtain $y = t$. However, note that this is not an equalibrium solution since it is not a constant. This is called the **horizontal isocline** since the target lines to the solution curves passing through this line is horizontal (have slope zero).

	$y < t$	$y > t$
y'	negative	positive
y	decreasing	increasing

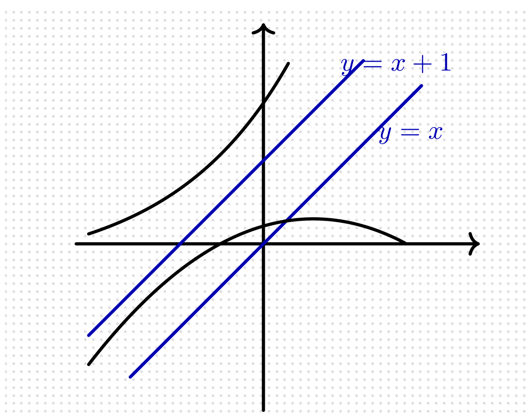
Again compute y'':

$$y'' = \frac{dy'}{dt} = \frac{d}{dt}(y - t) = y' - 1 = y - t - 1$$

which has a solution when $y = t + 1$:

	$y < t + 1$	$y > t + 1$
y''	negative	positive
y	down	up

We sketch the curve briefly:



□

Discovery 1.1. Why the curves are not crossing the line $y = t + 1$? This is because $y = t + 1$ is also a solution for the differential equation, and the solutions curves do not cross.

1.6 Existence and Uniqueness Theorem For First Order Differential Equations

Lecture 6 - Thursday, January 22

Theorem 1.3. Existence and Uniqueness Theorem

Consider the IVP $y' = f(x, y)$, $y(x_0) = y_0$. Suppose that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous at all points within a rectangle R:

$$R = \{(x, y) : |x - x_0| < a, \ |y - y_0| < b\},$$

where a and b are arbitrary constants. Then the IVP has a unique solution on some interval I containing x_0, within the length of the rectangle.

Example 1.22. Consider the IVP $y' = 3y^{2/3}$, $y(x_0) = 0$ for any $x_0 \in \mathbb{R}$. What does theorem 1.3 predict about the solution of the IVP?

Solution. We identify $f(y) = 3y^{2/3} = 3 \cdot \sqrt[3]{y^2}$, which is continuous for $y \in \mathbb{R}$. Compute

$$\frac{\partial f}{\partial y} = 3\left(\frac{2}{3}y^{2/3-1}\right) = \frac{2}{\sqrt[3]{y}}$$

which is not continuous for $y = 0$. For the initial condition $y(x_0) = 0$, and since $\frac{\partial f}{\partial y}$ is not continuous at $y = 0$, there is no prediction from theorem 1.3 because the conditions of the theorem are not satisfied. Let's check if we have a unique solution. we can obtain the general solution:

$$y(x) = (x + C_1)^3$$

Applying the initial condition $y(x_0) = 0$, we get

$$0 = (x_0 + C_1)^3 \quad \Rightarrow \quad C_1 = -x_0$$

which is any real number. Hence $y(x) = (x - x_0)^3$, and we can check $y' = 3\sqrt[3]{y^2}$. Notice that $y = 0$ is a trivial solution not included in the general solution. The solution to the IVP is not unique. □

Existence Uniqueness Theorem guarantees that the linear first order IVP (in the standard form)

$$y'(x) + p(x)y(x) = g(x), \qquad y(x_0) = y_0$$

has a unique solution if $p(x)$ and $g(x)$ are continuous on an open interval $\alpha < x < \beta$ containing x_0.

Comment 1.4. Alternatively, for

$$a_1(x)y'(x) + a_0(x)y(x) = f(x), \qquad y(x_0) = y_0$$

If $a_1(x)$, $a_0(x)$ and $f(x)$ are continuous on an open interval $\alpha < x < \beta$ and $a_1(x) \neq 0$ over that interval, and that interval contains x_0, then the Existence Uniqueness Theorem predicts a unique solution for the IVP.

Note 1.8. The largest interval that has no discontinuity for $a_1(x)$, $a_0(x)$ and $f(x)$ and $a_1(x) \neq 0$, containing x_0 is called interval of the solution (or the interval of validity of the solution).

Example 1.23. Use Existence Uniqueness Theorem to predict about the solution to the IVP, and determine the interval of solution

$$(x^2 - 1)\frac{dy}{dx} + xy = x^3, \qquad y(0) = 1$$

Solution. Identify the components in the IVP, we find that $a_1(x) = x^2 - 1$, which is continuous over $\mathbb{R}$; $a_0(x) = x$, which is continuous over $\mathbb{R}$; $f(x) = x^3$, which is again continous over $\mathbb{R}$. From condition 2, we need to exclude all points such that $a_1(x) = 0$. We find that there are two, $x = -1$ and $x = 1$. Therefore, the interval of solution is $(-1, 1)$. Finally, Existence Uniqueness Theorem 1.3 predicts that there is a uniqu solution for the IVP in $(-1, 1)$. $\square$

Exercise 1.7. Consider the IVP
$$x\frac{dy}{dx} + y^2 = 0, y(x_0) = 1$$

(a) Use Existence Uniquess Theorem to predict about the solution of the above IVP for (i) $x_0 = 1$, and (ii) $x_0 = 0$;

(b) Find the solution to the IVP with the initial condition $y(1) = 1$ and state the interval of the solution.

Exercise 1.8. Use Existence Uniqueness Theorem to predict the interval of the solution to the IVP:

$$(t - 2)y' + \ln(t + 1) \cdot y = te^t, \qquad y(0) = 2$$

1.7 Exact Differential Equations

Recall from multivariable calculus, suppose we have

$$f(x, y) = C \qquad y = y(x)$$

Differentiate both sides with respect to x we obtain

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = 0$$

Definition 1.16. **[Exact Differential Equation]**

Given differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

If $M(x,y)$ and $N(x,y)$ are contiuous on some open rectangle R and functions $f(x,y)$ is defined for all $(x,y) \in R$ and

$$M(x,y) = \frac{\partial f}{\partial x}, \qquad N(x,y) = \frac{\partial f}{\partial y}$$

Then we call the differential equation exact on R.

Example 1.24. The problem $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial x} = 0$ is exact and the solution is of the form $f(x,y) = C$ for some constant C.

1.7.1 Detecting Exact Differential Equations

If the DE is exact, then

$$M(x,y) = \frac{\partial f}{\partial x}, \qquad N(x,y) = \frac{\partial f}{\partial y}$$

Proposition 1.1. If both $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ exist and they are equal, i.e.,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

then the differential equation is exact.

1.7.2 Solution to Exact Differential Equations

Theorem 1.4. Existence Theorem for Exact DEs

If $M(x,y)$ and $N(x,y)$ and their partial derivatives M_y and N_x are all continuous on some open rectangle $R \subseteq \mathbb{R}^2$, suppose more that M_y and N_x agree on the rectangle, then the differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact on R, and there exists a twice differentiable function $f(x,y)$ on R with $f_x = M$, $f_y = N$, and the solution curves of the DE is $f(x,y) = C$.

Proof. Starting from the condition $M_y = N_x$ for exact DE. Starting from $M = \dfrac{\partial f}{\partial x}$, we have

$$f(x,y) = \int M \,\mathrm{d}x + g(y)$$

for some unknown function g of y. To determine $g(y)$, we take partial derivative with respect to y:

$$\frac{\partial f}{\partial y} = N(x,y) = \int \frac{\partial M}{\partial y}\,\mathrm{d}x + g'(y)$$

In particular, rearranging the equation, we obtain

$$g'(y) = N(x,y) - \int N_x(x,y)\,\mathrm{d}x$$

Solve for $g(y)$ by integrating with respect to y to get $g(y)$, so we know what $f(x,y) = \int M\,\mathrm{d}x + g(y)$. Rewrite the solution to obtain the form

$$f^1(x,y) = C$$

as desired.

Note 1.9. For this to be true, we must have

$$\frac{\partial}{\partial x}(g'(y)) = \frac{\partial}{\partial x}\left[N(x,y) - \int N_x(x,y)\,\mathrm{d}x\right] = 0$$

Notice that this is vacuously true, so as long as we find out what $g(y)$ is, we have the solution to the exact DE.

□

Comment 1.5. The theorem can also be proved by starting with a trial function $f(x,y) = \int N(x,y)\,\mathrm{d}y + h(x)$.

Example 1.25. Solve the IVP:

$$(2xy - 3x^2)dx + (x^2 - 2y)dy = 0, \qquad y(0) = 1$$

Solution. We first identify that

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

which implies that the differential equation is exact. Here we start with the function M:

$$\begin{aligned} f(x,y) &= \int (2xy - 3x^2)\,\mathrm{d}x + g(y) \\ &= yx^2 - x^3 + g(y) \end{aligned}$$

We know that

$$x^2 - 2y = N(x,y) = \frac{\partial f}{\partial y} = x^2 + g'(y)$$

so $g'(y) = -2y$. This implies that $g(y) = -y^2 + C$. Hence the general solution to the exact equation is

$$yx^2 - x^3 - y^2 = C$$

Applying the initial condition, we identify that the unknown constant is -1. Hence the solution to the IVP is

$$yx^2 - x^3 - y^2 = -1$$

as desired. $\square$

1.8 Special Non-linear First Order Differential Equations

1.8.1 Bernoulli Equation

Bernoulli equations are of the form

$$y'(x) + p(x)y(x) = g(x)y^r(x)$$

for some real number $r \in \mathbb{R}$.

1 Divide both sides by y^r:

$$\frac{1}{y^r}y' + p(x)\frac{1}{y^r}y = g(x)$$

2 Use the transformation $u(x) = y^{1-r}(x)$ and obtain

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = y'(1-r)y^{-r}$$

so $u' = (1-r)y^{-r}y'$

3 Substituting the above equation into the DE,

$$\frac{1}{1-r}u' + p(x)u = g(x)$$

which is our transformed DE.

4 Solve for $u(x)$ using the above linear first-order DE, and then we can find $y(x)$.

Algorithm 3: Procedure Solving Bernoulli Equations

Example 1.26. Solve the DE

$$y' = y\tan(x) - y^2\sec(x), \qquad x \in \left(0, \frac{\pi}{2}\right)$$

Solution. We first verify that this is a Bernoulli equation:

$$y' \underbrace{-\tan(x)}_{p(x)} y = \underbrace{-\sec(x)}_{g(x)} y^2$$

Dividing both sides by y^2, we get

$$y^{-2}y' - \tan(x)y^{-1} = -\sec(x)$$

Define $u(x) = y^{-1}(x)$, so $u' = y'(-1)y^{-2}$. Substituting $-u' = y^{-2}y'$ into the DE we get

$$-u'(x) - \tan(x)u(x) = -\sec(x)$$

which is equivalent to $u'(x) + \tan(x)u(x) = \sec(x)$, a linear first-order DE for $u(x)$. $\square$

Exercise 1.9. Solve the above linear first-order differential equation.

1.8.2 Ricatti Equation

Lecture 7 - Tuesday, January 27

Ricatti equations are of the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)y^2(x) + Q(x)y(x) + R(x)$$

Here is a systematic way solving Ricatti equations:

1. By inspection, we find a particular solution $y_p(x)$
2. We use a transformation $y(x) = z(x) + y_p(x)$
3. Substitute $y'(x)$ into the DE to find $z(x)$
4. The transformed DE is a Bernoulli equation (if not, check your work)
5. Solve the Bernoulli equation for $z(x)$
6. The general solution to the Ricatti equation is

$$y(x) = z(x) + y_p(x)$$

Algorithm 4: Procedure Solving Ricatti Equations

Example 1.27. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3(y - x)^2 + \frac{y}{x}$$

Proof. We rewrite the differential equation to obtain the following equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3y^2 + \left(\frac{1}{x} - 2x^4\right)y + x^5$$

which is a Ricatti equation. By inspection, we notice that $y(x) = x$ is a solution. Hence our final solution would be in the form of

$$y(x) = z(x) + x$$

Substitute $y' = z' + 1$ into the DE we get

$$z' + 1 = x^3(z + x - x)^2 + \frac{z + x}{x} \quad \Rightarrow \quad z' = x^3z^2 + \frac{z}{x}$$

which is a Bernoulli equation. Solving for z, we would get

$$z(x) = \frac{1}{\frac{C}{x} - \frac{1}{5}x^4}$$

Hence our general solution to the DE is $y(x) = \dfrac{1}{\frac{C}{x} - \frac{1}{5}x^4} + x$. $\square$

1.8.3 The Method of Variation of Parameters (to find $y_p(x)$, for Linear 1st-order DEs)

We establish the procedure here:

1. Standard form is
$$y'(x) + p(x)y(x) = g(x)$$
2. Find the homogeneous solution to $y'(x) + p(x)y(x) = 0$, $y_h(x) = Ce^{-\int p(x)\,\mathrm{d}x}$
3. $y_p(x) = c(x)e^{-\int p(x)\,\mathrm{d}x}$, we take $y_h(x)$ and replace the arbitrary constant with a function of x, substituting $y_p(x)$ into the DE to find $c(x)$.
4. Once we have $c(x)$, we have $y_p(x)$
5. The general solution to the linear first-order DE is
$$y(x) = y_h(x) + y_p(x)$$
In other words, we have $y(x) = Ce^{-\int p(x)\,\mathrm{d}x} + c(x)e^{-\int p(x)\,\mathrm{d}x}$

Algorithm 5: The Method of Variation of Parameters

Example 1.28. Solve the differential equation using the method of variation of parameters
$$y' + 2y = e^{-x}$$

Proof. Find the homogeneous solution, we easily get $y_h(x) = Ce^{-2x}$. Therefore, we have
$$y_p(x) = c(x)e^{-2x} \quad \Rightarrow \quad y_p' = c'e^{-2x} - 2ce^{-2x}$$
Substituting it into the DE, we obtain
$$c'e^{-2x} - 2ce^{-2x} + 2(ce^{-2x}) = e^{-x}$$
Solving for $c(x)$ we get
$$c(x) = \int e^x\,\mathrm{d}x + C_1 = e^x + C_1 \quad \Rightarrow \quad y_p(x) = (e^x + C_1)e^{-2x}$$
Therefore, the general solution to the DE is given by
$$\begin{aligned} y(x) &= Ce^{-2x} + e^x e^{-2x} + C_1e^{-2x} \\ &= C_2e^{-2x} + e^{-x} \end{aligned}$$
as desired. □

1.9 Exercises

1.9.1 Linear

Exercise 1.10. Solve the IVP:

$$\cos(x)\, y' + \sin(x)\, y = 2\cos^3(x)\sin(x) - 1, \qquad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}, \qquad 0 \le x < \frac{\pi}{2}.$$

Exercise 1.11. Find the solution to the IVP:

$$t\, y' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2}.$$

Exercise 1.12. Find the solution to the IVP:

$$t\, y' - 2y = t^5 \sin(2t) - t^3 + 4t^4, \qquad y(\pi) = \frac{3}{2}\pi^4.$$

Exercise 1.13. Find the solution to the IVP and determine all possible behaviors of the solution as $t \to \infty$. If this behavior depends on the value of y_0, give this dependence:

$$2y' - y = 4\sin(3t), \qquad y(0) = y_0.$$

1.9.2 Separable

Exercise 1.14. Solve the following differential equation and determine the interval of validity for the solution:

$$\frac{dy}{dx} = 6y^2 x, \qquad y(1) = \frac{1}{25}.$$

Exercise 1.15. Solve the following IVP and find the interval of validity for the solution:

$$y' = \frac{3x^2 + 4x - 4}{2y - 4}, \qquad y(1) = 3.$$

Exercise 1.16. Solve the following IVP and find the interval of validity of the solution:

$$y' = \frac{xy^3}{\sqrt{1 + x^2}}, \qquad y(0) = -1.$$

Exercise 1.17. Solve the following IVP and find the interval of validity of the solution:

$$y' = e^{-y}(2x - 4), \qquad y(5) = 0.$$

Exercise 1.18. Solve the following IVP and find the interval of validity for the solution:

$$\frac{dr}{d\theta} = \frac{r^2}{\theta}, \qquad r(1) = 2.$$

Exercise 1.19. Solve the following IVP:

$$\frac{dy}{dt} = e^{\,y-t}\sec(y)\,(1+t^2), \qquad y(0) = 0.$$

1.9.3 Exact

Exercise 1.20. Solve the following IVP and find the interval of validity for the solution:

$$2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0, \qquad y(0) = -3.$$

Exercise 1.21. Find the solution and interval of validity for the following IVP:

$$2xy^2 + 4 = 2\left(3 - x^2 y\right) y', \qquad y(-1) = 8.$$

Exercise 1.22. Find the solution and interval of validity to the following IVP:

$$\frac{2ty}{t^2+1} - 2t - \left(2 - \ln(t^2+1)\right) y' = 0, \qquad y(5) = 0.$$

Exercise 1.23. Find the solution and interval of validity for the following IVP:

$$3y^3 e^{3xy} - 1 + \left(2ye^{3xy} + 3xy^2 e^{3xy}\right) y' = 0, \qquad y(0) = 1.$$

1.9.4 Bernoulli

Exercise 1.24. Solve the following IVP and find the interval of validity for the solution:

$$y' + \frac{4}{x} y = x^3 y^2, \qquad y(2) = -1, \qquad x > 0.$$

Exercise 1.25. Solve the following IVP and find the interval of validity for the solution:

$$y' = 5y + e^{-2x} y^{-2}, \qquad y(0) = 2.$$

Exercise 1.26. Solve the following IVP and find the interval of validity for the solution:

$$6y' - 2y = xy^4, \qquad y(0) = -2.$$

Exercise 1.27. Solve the following IVP and find the interval of validity for the solution:

$$y' + \frac{y}{x} - \sqrt{y} = 0, \qquad y(1) = 0.$$

1.9.5 Ricatti

Exercise 1.28. Riccati IVP 1 (constant coefficients).

$$y' = y^2 - 4y + 3, \qquad y(0) = 2.$$

Exercise 1.29. Riccati IVP 2 (polynomial coefficients).

$$y' = y^2 + x\,y + x^2, \qquad y(0) = 0.$$

Exercise 1.30. Riccati IVP 3 (with an easy particular solution).

$$y' = y^2 + \left(\frac{1}{x} - 2x\right) y + (x^2 - 1), \qquad y(1) = 1, \quad x > 0.$$

Exercise 1.31. Riccati IVP 4 (trigonometric coefficients).

$$y' = (\sin x)\, y^2 + (2\cos x)\, y + \sin x, \qquad y\left(\frac{\pi}{2}\right) = 0.$$

2 Dimensional Analysis

Here are some notation conventions, or fundamental dimensions. We use L to denote the dimension of length, T for the dimension of time, M for mass, Q for charge, and K for temprature.

Example 2.1. Consider velocity, we know that

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \Rightarrow \quad [v] = \frac{[x]}{[t]} = \frac{L}{T} = LT^{-1}$$

Comment 2.1. Some parameters have dimension 1 or even dimension-less.

Example 2.2. The sine of any angle θ has dimension 1. By linearization, we know that

$$\sin\theta \approx \theta$$

so as a result, θ has dimension 1.

2.1 Dimensional Principles (DP)

Theorem 2.1. Dimensional Principle 1 — Dimensional Homogenity

For equations (equalities) or inequalities with meaningful physical contents, only terms with the same dimension can be added, subtracted, equated, or compared. i.e., for instance, we have inequality

$$A \pm B \leq C$$

only if $[A] = [B] = [C]$.

Theorem 2.2. Dimensional Principle 2

For quantities A and B having different dimensions, we can only multiply or divide, and

$$[AB] = [A][B], \qquad \left[\frac{A}{B}\right] = \frac{[A]}{[B]}$$

Example 2.3. Use the DE for the falling object near the surface of the Earth with no drag force:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{MG}{(R+s)^2}$$

to find the dimension of Newton's universal gravitation constant, G. By the way,

- v is velocity
- t is time
- M is mass of the Earth
- R is radius of the Earth
- s is the distance of the object from the surface of the Earth

Proof. By dimensional principle 1 (2.1), we have

$$\left[\frac{\mathrm{d}v}{\mathrm{d}t}\right] = \left[-\frac{MG}{(R+s)^2}\right]$$

By dimensional principle 2 (2.2), we know that

$$\frac{[v]}{[t]} = \frac{[M][G]}{[R+s]^2}$$

Hence $[G] = L^3T^{-2}M^{-1}$. $\square$

2.2 Dimensionless Variables

Sometimes the choice of suitable dimensionless variables simplifies the DE and its solution for known models.

Example 2.4. For a projectile motion: If we ignore air resistance, the displacement $s(t)$ of a projectile launched vertically at $t = 0$ from the initial height s_0 with initial velocity v_0 satisfies the IVP.

$$F = ma \qquad \Rightarrow \qquad -mg = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

and hence the velocity is given by $v(t) = -gt + v_0$. Furthermore, we get

$$s(t) = -\frac{1}{2}gt^2 + v_0t + C_2$$

Using the initial condition to solve for C_2 we get

$$s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$$

Lecture 8 - Thursday, January 29

To write the IVP in dimensionless form, we define $\tilde{s} = \frac{s(t)}{s_C}$ where s_C is characteristic distance and is a constant with the same dimension of S, so $\tilde{s}$ is dimensionless. We observe that $s_C = s_0$. Similarly, we define $\tilde{t} = \frac{t}{t_C}$ where t_C is characteristic time and is a constant with the same dimension of time, so $\tilde{t}$ is also dimensionless. Substituting them into out equation we obtain

$$\tilde{s}s_C = -\frac{1}{2}g(\tilde{t}t_C)^2 + v_0(\tilde{t}t_C) + s_0$$
$$\tilde{s}(\tilde{t}) = -\frac{1}{2}g\frac{t_C^2}{s_C}\tilde{t}^2 + \frac{v_0t_C}{s_C}\tilde{t} + \frac{s_0}{s_C}$$

We can easily verify that the last two terms are dimensionless. LHS is also dimensionless, so we must have the term $g\frac{t_C^2}{s_C}$ being dimensionless. Define $\alpha = \frac{v_0}{s_0}\sqrt{\frac{s_0}{g}}$, which can be verified is dimensionless, then we can simplify the equation to

$$\tilde{s}(\tilde{t}) = -\frac{1}{2}\tilde{t}^2 + \alpha\tilde{t} + 1$$

Example 2.5. Alternatively, we can write $s(t)$ in dimensionless form by first inspect that $s_C = s_0$. For $t_C = \dfrac{v_0}{g}$, we have

$$[t_C] = \frac{LT^{-1}}{LT^{-2}} = T$$

We define

$$\tilde{t} = \frac{t}{t_C} = \frac{t}{v_0/g}, \qquad \tilde{s} = \frac{s}{s_C} = \frac{s}{s_0}$$

Substituting this into the equation we had, we get

$$\begin{aligned}\tilde{s}s_0 &= -\frac{1}{2}g\tilde{t}^2\left(\frac{v_0}{g}\right)^2 + v_0\tilde{t}\left(\frac{v_0}{g}\right) + s_0\\ \tilde{s}(\tilde{t}) &= -\underbrace{\frac{1}{2}\left(\frac{{v_0}^2}{gs_0}\right)}_{\lambda}\tilde{t}^2 + \tilde{t}\underbrace{\frac{{v_0}^2}{gs_0}}_{2\lambda} + 1\end{aligned}$$

Verify $[\lambda] = \frac{[v_0]^2}{[g][v_0]} = \frac{L^2T^{-2}}{LT^{-2}L} = 1$, so we have a dimensionless equation:

$$\tilde{s}(\tilde{t}) = -\lambda\tilde{t}^2 + 2\lambda\tilde{t} + 1$$

Note 2.1. Both forms of $\tilde{s}(\tilde{t})$ are dimensionless and valid:

$$\begin{cases}\tilde{s}(\tilde{t}) = -\frac{1}{2}\tilde{t}^2 + \alpha\tilde{t} + 1\\ \tilde{s}(\tilde{t}) = -\lambda\tilde{t}^2 + 2\lambda\tilde{t} + 1\end{cases}$$

We have simplied $s(t)$ with three dimensional constants to a dimensionless form with only one dimensionless constant.

Comment 2.2. For $v(t) = -gt + v_0$. At the maximum height, we have $v(t) = 0$, which givens us

$$0 = -gt + v_0 \quad \Rightarrow \quad t_C = \frac{v_0}{g}$$

which is what we defined above. Moreover, for the λ defined above,

$$\lambda = \frac{1/2m{v_0}^2}{mgs_0} = \frac{\text{initial kinetic energy}}{\text{initial potential energy}}$$

Example 2.6. Consider the simple model for population growth:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = rp - k, \qquad p(0) = p_0$$

where r and k are constant. Write the IVP in dimensionless form, and find the dimensions of r and k.

Solution. To find the dimensions of unknown constants, we use DP1 (2.1) and DP2 (2.2): We first know from DP1 (2.1) that:

$$\left[\frac{\mathrm{d}p}{\mathrm{d}t}\right] = [rp] = [k]$$

and hence

$$\frac{[p]}{[t]} = [r][p] \quad \Rightarrow \quad [r] = T^{-1}$$
$$\frac{[p]}{[t]} = [k] \quad \Rightarrow \quad [k] = PT^{-1}$$

We define

$$\tilde{p} = \frac{p}{p_C} \quad \Rightarrow \quad p_C = p_0$$
$$\tilde{t} = \frac{t}{t_C} \quad \Rightarrow \quad t_C = \frac{1}{r}$$

so we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{p}p_C) = p_0\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{t}}\left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}\right) = p_0 r\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{t}}$$

Substituting this back into the differential equation, we get

$$p_0 r\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{t}} = r\tilde{p}p_0 = k \quad \Rightarrow \quad \frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{t}} = \tilde{p} - \lambda$$

where $\lambda = k/(rp_0)$, which is dimensionless. For initial condition $p(0) = p_0$, we get

$$p_0\tilde{p}(0) = p_0 \quad \Rightarrow \quad \tilde{p}(0) = 1$$

Therefore, the dimensionless IVP is

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{t}} = \tilde{p} - \lambda, \qquad \tilde{p}(0) = 1$$

as desired. □

2.2.1 Finding the Dimension Ratios

Here is the question, given a set of dimensional quantities (parameters): $(Q_1, Q_2, \ldots, Q_n)$, how can we find the possible dimensionless quantities from them?

BY DP2 (2.2), we essentially solve for constants $c_i \in \mathbb{R}$ such that

$$\pi = Q_1^{c_1} Q_2^{c_2} \cdots Q_n^{c_n}$$

is dimensionless, i.e., $[\pi] = 1$.

Example 2.7. From example for projectile motion no drag force:

$$\lambda = \frac{\frac{1}{2}mv_0^2}{mgs_0}$$

for physical quantities (m, v_0, g, s_0). We wish to have that for R defined as:

$$\pi = m^{c_1} g^{c_2} s_0^{c_3} v_0^{c_4}$$

$[\pi] = 1$. Equivalently, we wish

$$\begin{aligned} [m]^{c_1}[g]^{c_2}[s_0]^{c_3}[v_0]^{c_4} &= 1 \\ M^{c_1}(LT^{-2})^{c_2}(L)^{c_3}(LT^{-1})^{c_4} &= 1 \\ M^{c_1}L^{c_2+c_3+c_4}T^{-2c_2-c_4} &= 1 \end{aligned}$$

Solve for

$$\begin{cases} c_1 = 0, \\ c_2 + c_3 + c_4 = 0 \;\Rightarrow\; c_3 = -c_4 - c_2, \\ -2c_2 - c_4 = 0 \;\Rightarrow\; c_4 = -2c_2. \end{cases}$$

$$c_3 = (2c_2) - c_2 = c_2.$$

so c_2 is the free parameter. If $c_2 = -1$, then we have

$$\pi = \frac{v_0^2}{gs_0}$$

which is what we had before.

Dimensional Matrix Another way of solving the above problem is to use the dimensional matrix. In particular, solving the system of equations

$$\begin{cases} c_1 + 0 + 0 + 0 = 0, \\ 0 + c_2 + c_3 + c_4 = 0, \\ 0 - 2c_2 + 0 - c_4 = 0, \end{cases}$$

is equivalent to solving

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving $D\vec{c} = \vec{0}$ we find a dimensionless quantity.

Discovery 2.1. Alternatively, we can write dimensional matrix D from the dimensions of quantities:

$$[m] = M, \qquad [v_0] = LT^{-1}, \qquad [s_0] = L, \qquad [g] = LT^{-2}.$$

Therefore, the dimensional matrix is:

$$D = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccc} m & g & s_0 & v_0 \end{array}$$
$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \end{pmatrix} \begin{array}{c} M \\ L \\ T \end{array}$$

Lecture 9 - Tuesday, February 03

2.2.2 Buckingham Pi Theorem

Theorem 2.3. Buckingham Pi Theorem

The number of independent dimensionless quantities is the number of physical quantities involved in the problem minus the rank of the dimensional matrix.

Example 2.8. For the projectile motion (no drag) problem, there are four physical quantities (m, g, v_0, s_0), and the rank of the dimensional matrix is 3. Hence the number of dimensionless quantities is $4 - 3 = 1$.

Note 2.2. If we have $(n - r)$ dimensionless quantities, then we have to find dimensionless quantites

$$\pi_1, \pi_2, \ldots, \pi_{n-r}$$

If $n - r = 1$, then π is a constant. Else if $n - r \neq 1$, then we can write one of these dimensionless quantitesas as a function of the rest.

Example 2.9. Consider the population growth problem, recall

$$\frac{\mathrm{d}p}{\mathrm{d}t} = rp - k, \qquad p(0) = p_0$$

Assuming that the population p is measured by biomass and write the dimensionless.

Solution. We have $[p] = M$, $[t] = T$, $[k] = MT^{-1}$, $[r] = T^{-1}$, and $[p_0] = M$. Notice that we have 5 physical quantities, and the rank of the dimension matrix D,

$$D = \begin{array}{cc} & \begin{array}{ccccc} P & t & k & r & p_0 \end{array} \\ & \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 \end{pmatrix} \begin{array}{c} M \\ T \end{array} \end{array}$$

has rank of 2. Hence the number of dimensionless quantities is 3. We can find these dimensionless quantitesas either by solving $D\vec{c} = \vec{0}$ or by inspection. By inspection (this is not unique),

$$\pi_1 = \frac{pr}{k}, \qquad \pi_2 = rt, \qquad \pi_3 = \frac{k}{rp_0}$$

We can verify that π_1, π_2, π_3 are dimensionless. □

Exercise 2.1. For the above example

(a) Solve $D\vec{c} = \vec{0}$ and find π_1, π_2, and π_3.

(b) Using dimensionless variables $\tilde{p} = p/(k/r)$ and $\tilde{t} = rt$, show that the IVP can be written as

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{t}} = \tilde{p} - 1, \qquad \tilde{p}(0) = \frac{rp_0}{k}$$

in dimensionless form.

2.2.3 Using Dimensional Analysis to Explore an Unknown Parameters

We can use

$$Q_1 = f(Q_2, Q_3, \ldots, Q_n)$$

where Q_is are dimensional parameters. Similarly, by Buckingham Pi Theorem, we have

$$\pi_1 = f(\pi_2, \ldots, \pi_{n-r})$$

where π_is are dimensionless quantities.

Note 2.3. Quantites Q_1 and π_1 are chosen arbitrarily.

Example 2.10. Suppose we wish to apply the Buckingham Pi Theorem to discover how the terminal velocity V_T in case of Newtonian drag force βv^2 is related to the other physical parameters like m, g, and β in downward motion of sky diving with mass of m. That is, without even formulating a model (DE), we wish to determine $V_T = f(m, g, \beta)$ as far as possible.

Solution. We have the following physical parameters:

$$[m] = M, \quad [g] = LT^{-2}, \quad [\beta], \quad [V_T] = LT^{-1}$$

We know that the drag force is βv^2, and we know that the dimension of force is MLT^{-2}, so $[\beta] = ML^{-1}$. The dimensional matrix is thus

$$D = \begin{array}{c} \begin{matrix} V_T & m & g & \beta \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -2 & 0 \end{pmatrix} \end{array} \begin{matrix} M \\ L \\ T \end{matrix}$$

which has rank of 3, so the number of dimensionless quantities is $4 - 3 = 1$. By inspection,

$$[\pi] := \left[\frac{V_T^2 \beta}{mg}\right] = 1$$

which is dimensionless. Recall Buckingham Pi Theorem, since $n - r = 1$, so π we found is a constant, say C. This tells us that

$${V_T}^2 = C \cdot \frac{mg}{\beta} \quad \Rightarrow \quad V_T = \pm\sqrt{C}\sqrt{\frac{mg}{\beta}} \quad \Rightarrow \quad V_T = C'\sqrt{\frac{mg}{\beta}}$$

$\square$

Example 2.11. Consider a simple pendulum with mass of m and length of ℓ, released from rest at initial angle θ_0. Assuming no drag force in the motion, the only force acting on the system is gravity, mg. Use Buckingham Pi Theorem to find the relation between the period of the pendulum and other physical parameters involved.

Solution. Again, we first identify what we have:

$$[T] = T, \quad [m] = M \quad [\ell] = L, \quad [g] = LT^{-2}, \quad [\theta_0] = 1$$

The dimensional matrix is given by

$$D = \begin{array}{c} \begin{array}{ccccc} T & m & \ell & g & \theta_0 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ M \\ L \\ T \end{array}$$

The rank of the matrix is 3, so the number of dimensionless quantities we expect is $5 - 3 = 2$. By inspection, we can take

$$\pi_1 = \theta_0, \qquad \pi_2 = \frac{T^2 g}{\ell}$$

By Buckingham Pi theorem, we know that π_2 is a function of π_1: $\pi_2 = F(\pi_1)$ for some F. Hence

$$\frac{T^2 g}{\ell} = F(\theta_0) \quad \Rightarrow \quad T = \sqrt{F(\theta_0)}\sqrt{\frac{\ell}{g}}$$

where $\sqrt{F(\theta_0)}$ is just a constant. $\square$

Example 2.12. Consider an object moving through a Newtonian fluid. We assume that the density of the object is much larger than the density of surrrounding fluid so buoyancy and added mass forces can be ignored. The governing equation is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg + F_d(v)$$

where the drag force F_d depends on the object velocity and shape. We assume the object is rotationally symmetric about the vertical axis and is not spinning, so the drag force acts to oppose the object motion. That means that there is no tangential component of the force. The drag force depends on the fluid density ρ_f, the size of the object given by, for example a cross section diameter D, the object velocity v, and the fluid viscosity μ ($ML^{-1}T^{-2}$). What are the dimensionless quantities? Find F_d as a function of the other physical quantities.

Solution. Identify what we know:

$$[F_d] = MLT^{-2}, \quad [\rho_f] = ML^{-3}, \quad [D] = L, \quad [\mu] = ML^{-1}T^{-1}, \quad [v] = LT^{-1}$$

The dimensional matrix is given as

$$D = \begin{array}{c c} & \begin{array}{ccccc} F_d & \rho_f & D & \mu & v \end{array} \\ & \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & -3 & 1 & -1 & 1 \\ -2 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{array}{c} M \\ L \\ T \end{array} \end{array}$$

The rank of the matrix is 3, so there will be $5 - 3 = 2$ dimensionless quantities. By inspection,

$$\pi_1 = \frac{F_d}{\frac{1}{2}\rho_f D^2 v^2}, \qquad \pi_2 = \frac{vD\rho_f}{\mu}$$

where in π_1, the factor of $1/2$ in the denominator is traditional, it does not effect our answer, it is there to be precise. By Buckingham Pi theorem, we know $\pi_1 = F(\pi_2)$. Hence

$$F_d = \frac{1}{2} \cdot F\left(\frac{vD\rho_f}{\mu}\right) \cdot \rho_f D^2 v^2$$

□

Exercise 2.2. A spherical object is moving through air experiences a drag force that slows down the motion. The drag force, F, depends on the velocity, spherical density ρ, the radius r, and the drag coefficient C_A which is dimensionless. Find an expression for the drag force using Buckingham Pi theorem and these physical quantities. How does the drag force scale with v and r.

3 Higher Order Differential Equations

Lecture 10 - Thursday, February 05

3.1 Reduction of Order

3.1.1 Dependent Variable is Missing

This is the scenario that we only have y'., y'' and/ir $g(x)$ but $y(x)$ is missing in the differential equation. Here is what we are going to do

1 $u(x) = y'(x)$, transformaton: reducing the order of DE
2 $u'(x) = y''(x)$, substituting into the DE, now we have a 1st order DE in terms of $u(x)$
3 Solve the 1st order DE and return the original variable via integration.

Algorithm 6: What to do when Dependent Variable is missing

Example 3.1. Solve for $y'' + y' = t$.

Solution. We solve this via reduction of order, define $u = y'$. Hence we obtain the DE:

$$u' + u = t$$

where we know the solution (obtained using integrating factor, or undetermined coefficient/variation of parameters is also fine) is

$$u(t) = t - 1 + Ce^{-t}$$

Hence the solution y is $y(t) = \frac{1}{2}t^2 - t - Ce^{-t} + C'$. □

3.1.2 Independent Variable is Missing

In this case, we know know a relation between y', y'', and y. See below for an example.

Example 3.2. Solve $y'' = 2y(y')^3$, $y = y(x)$.

Solution. This is a non-linear 2nd order DE missing the independent variable. We first define $y' = u$, so

$$y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}u}{\mathrm{d}y} = 2y(u)^3$$

Assuming $u \neq 0$, we can solve for u: $u(y) = \frac{1}{C-y^2}$, and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{C - y^2}$$

Now we can solve this and obtain

$$Cy - \frac{y^3}{3} = x + C'$$

which is an implicit solution. Recall that we made the assumption that $u \neq 0$, if $u = 0$, then $y(x) = C''$ is some constant, which is not included in the implicit form. This is our exceptional solution. Therefore, the solution to the 2nd order DE is

$$Cy - \frac{y^3}{3} = x + C' \qquad \text{and} \qquad y(x) = C''$$

as desired. $\square$

3.2 Higher Order Differential Equations

Higher order differential equations are of the form

$$F(x, y, y', y'', \dots, y^{(n)}) = 0, \qquad y = y(x)$$

Definition 3.1. **[Linearity]**

Again, a differential equation is **linear** if F is a linear function of $y, y', y'', \dots, y^{(n)}$.

Given a differential equation of the form

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)}(x) + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

- If $f(x) = 0$, then this is a homogeneous DE;
- If $f(x) \neq 0$, then this is a inhomogeneous DE;
- If all $a_i(x)$ are constant, then this is a constant coefficient DE

Comment 3.1. For IVP of nth order, we need n initial conditions:

$$y(x_0) = p_0, \quad y'(x_1) = p_1, \quad \dots, \quad y^{(n-1)}(x_{n-1}) = p_{n-1}$$

where p_i are constants for all i.

3.2.1 Existence and Uniqueness Theorem Revisited

Theorem 3.1. Existence and Uniqueness Theorem

For the IVP

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)}(x) + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

$$y(x_0) = p_0, \quad y'(x_0) = p_1, \quad \dots, \quad y^{(n-1)}(x_0) = p_{n-1}$$

If there is an open interval I such that:

1. all functions $a_i(x)$ and $f(x)$ are continous on I;
2. $a_n(x) \neq 0$ on I;
3. for any $x_0 \in I$,

then the there is a unique solution on I and I is the interval of the solution.

Example 3.3. Consider the IVP:

$$y'' + \frac{y'}{x-3} + \sqrt{x}y = \ln x, \qquad y(1) = 3, y'(1) = -5$$

Find the interval of solution using the E/U theorem 3.1.

Solution. We identify:

$$a_2(x) = 1, \quad a_1(x) = \frac{1}{x-3}, \quad a_0(x) = \sqrt{x}, \quad f(x) = \ln x$$

so the interval(s) that all a_is and f are continous is

$$(0,3) \cup (3, +\infty)$$

Therefore, the interval of solution is $(0, 3)$. □

3.2.2 The Principle of the Superposition Theorem

Theorem 3.2. The Principle of the Superposition Theorem

For any linear n-order DE:

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)}(x) + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

If $y_1(x), y_2(x), \ldots, y_n(x)$ are the solutions of the homogeneous DE

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)}(x) + \cdots + a_1(x)y' + a_0(x)y = 0$$

on I, then for any constant $C_1, C_2, \ldots, C_n \in \mathbb{R}$,

$$y_h(x) = C_1y_1(x) + \cdots + C_ny_n(x)$$

is a solution to the homogeneous DE.

Definition 3.2. **[Linearly Dependent]**

The functions $y_1(x)$, $y_2(x), \ldots, y_n(x)$ are said to be **linearly dependent** on an open interval I if there exists constants $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that not all α_is are zero at the same time, and

$$\alpha_1y_1 + \alpha_2y_2 + \cdots + \alpha_ny_n = 0$$

Definition 3.3. **[Linearly Independent]**

The functions $y_1(x)$, $y_2(x), \ldots, y_n(x)$ are said to be **linearly independence** if they are not linear dependent.

Example 3.4. Are the functions e^x, e^{-x} and $\sinh(x)$ linear dependent?

Solution. They are linear dependent. □

3.3 Wronskian and Linear Independence

Definition 3.4. **[Wronskian]**

The **wronskian** of n differential functions

$$y_1(x), y_2(x), \ldots, y_n(x)$$

is given by

$$W[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$

Example 3.5. For $W[e^x, e^{-x}]$, we know

$$W[e^x, e^{-x}] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Proposition 3.1. Let $y_1(x), y_2(x), \ldots, y_n(x)$ be n solutions on an open interval I of the DE:

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)}(x) + \cdots + a_1(x)y' + a_0(x)y = 0$$

Then $y_1(x), y_2(x), \ldots, y_n(x)$ are linear independence on I if $W[y_1, y_2, \ldots, y_n] \neq 0$ for all $x \in I$.

3.3.1 Theorem for a 2nd Order Linear Differentiate Equation

Theorem 3.3.

In the standard form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

If the conditions of the Existence Uniqueness theorem 3.1 are satisfied and $y_1(x)$ and $y_2(x)$ are linearly independent solution on the same interval $\alpha < x < \beta$, then for any $x_0 \in (\alpha, \beta)$ and any constant p_0, p_1, there exist unique constants C_1 and C_2 such that

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x)$$

is a solution to the IVP

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \qquad y(x_0) = p_0, \;\; y'(x_0) = p_1$$

3.3.2 Abel's Formula for a 2nd Order Differential Equation

Lemma 3.1. If $y_1(x)$ and $y_2(x)$ are any two solutions of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

on I, then the wronskian is either zero or non-zero on all of I.

Note 3.1. In particular, Abel's formula tells us that

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

is in the form of

$$W(x) = Ce^{-\int p(x)\,\mathrm{d}x}$$

Proof. We start from the solutions: Suppose y_1, y_2 are solutions of

$$y'' + p(x)\,y' + q(x)\,y = 0.$$

That is,

$$\begin{cases} y_1'' + p(x)\,y_1' + q(x)\,y_1 = 0, \\ y_2'' + p(x)\,y_2' + q(x)\,y_2 = 0. \end{cases}$$

We know that the wronskian is defined as

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

Differentiate:

$$W' = (y_1 y_2' - y_2 y_1')' = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' = y_1 y_2'' - y_2 y_1''.$$

From the differential equation (applied to y_1 and y_2),

$$y_1'' = -p(x)\,y_1' - q(x)\,y_1, \qquad y_2'' = -p(x)\,y_2' - q(x)\,y_2.$$

Substitute into W':

$$\begin{aligned} W' &= y_1(-p(x)y_2' - q(x)y_2) - y_2(-p(x)y_1' - q(x)y_1) \\ &= -p(x)y_1y_2' - q(x)y_1y_2 + p(x)y_2y_1' + q(x)y_2y_1 \\ &= -p(x)\big(y_1y_2' - y_2y_1'\big) \\ &= -p(x)\,W. \end{aligned}$$

Hence $W' + p(x)\,W = 0$. Solving this first-order linear ODE gives

$$W(x) = C\,e^{-\int p(x)\,dx}, \qquad \text{where } C = W(x_0),\ x_0 \in I.$$

Since $e^{-\int p(x)\,dx}$ exists for continuous $p(x)$ and is never 0, we conclude $W(x)$ is either identically 0 (if $C = 0$) or never 0 (if $C \neq 0$). $\square$

Example 3.6. Consider $\phi_1(x) = x^2 - 2x$, $\phi_2(x) = -1$ on $(-2, 2)$. What is your conclusion about ϕ_1 and ϕ_2? Are they solutions to a 2nd order DE?

Solution. We compute the wronskian:

$$W[x^2 - 2x, -1] = 2x - 2$$

on $(-2, 2)$. We observe that $W(x_0 = 0) = -2$ and $W(x_0 = 1) = 0$, so the values of wronskian changes from non-zero to zero, so we conclude that these functions are not solutions to a 2nd order DE. $\square$

3.4 General Solution to Inhomogenous Linear Differential Equations

Lecture 11 - Tuesday, February 10

Suppose we have the inhomogeneous linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)}(x) + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

1. Find the homogeneous solution to the homogeneous differential equation:

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)}(x) + \cdots + a_1(x)y' + a_0(x)y = 0$$

   First obtain the solutions $y_1(x), y_2(x), \ldots, y_n(x)$. They need to be such that $W[y_1, y_2, \ldots, y_n] \neq 0$ (i.e., linear independence) for all $x \in I$, and the homogeneous solution is

$$y_h = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

2. Find a particular solution $y_p(x)$;
3. The general solution is $y(x) = y_h(x) + y_p(x)$.

3.4.1 Linear 2nd Order Differential Equation with Constant Coefficients

Linear 2nd order differential equation with constant coefficients are of the form

$$A_2 y''(x) + A_1 y'(x) + A_0 y(x) = 0$$

We first write it in the standard form

$$y'' + \frac{A_1}{A_2} y'(x) + \frac{A_0}{A_2} y(x) = 0$$

which is of the form

$$y'' + py' + qy = 0$$

Example 3.7. [Case $p = 0$, $q = -1$]: In this case, the differential equation is of the form

$$y'' - y = 0$$

Notice that $y(x) = e^{\pm x}$ are two solutions to the differential equation. Moreover,

$$W[e^x, e^{-x}] = -2$$

which implies that the two solutions are linearly independent. Therefore, the solution to the above DE is

$$y_h(x) = C_1 e^x + C_2 e^{-x}$$

In general, for DE

$$y'' + py' + qy = 0$$

Assume $y(x) = e^{\lambda x}$ is a solution, then we must have (by substituting in the solution)

$$\lambda^2 + p\lambda + q = 0$$

which is called the *characteristic equation*. We know that the two solutions are $\lambda_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$. Now we have three cases depends on the value of the discriminant:

[Case 1, $p^2 - 4q > 0$**]:** In this case, there are two distinct roots,

$$\lambda_1 = \frac{-p + \sqrt{p^2 - 4q}}{2}, \qquad \lambda_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}$$

and the two solutions to the original DE are

$$y_1(x) = e^{\lambda_1 x}, \qquad y_2(x) = e^{\lambda_2 x}$$

Checking if the two solutions are linearly independent:

$$W[e_1^{\lambda} x, e^{\lambda_2 x}] = (\lambda_1 - \lambda_2) e^{(\lambda_1 + \lambda_2)x}$$

We verify that they are linearly independent, so the homogeneous solution is

$$\boxed{y_h(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}}$$

[Case 2, $p^2 - 4q < 0$**]:** In this case, we have two complex conjugate roots

$$\lambda_1 = \frac{-p + i\sqrt{4q - p^2}}{2}, \qquad \lambda_2 = \frac{-p - i\sqrt{4q - p^2}}{2}$$

and the two solutions to the original DE are

$$y_1(x) = e^{\lambda_1 x} = e^{A + iB}, \qquad y_2(x) = e^{\lambda_2 x} = e^{A - iB}$$

We are interested in real solutions, manipulate the solutions

$$\begin{aligned} y_1(x) &= e^{Ax}e^{iBx} \\ &= e^{Ax}\cos(Bx) + ie^{Ax}\sin(Bx) \\ y_2(x) &= e^{Ax}\cos(Bx) - ie^{Ax}\sin(Bx) \end{aligned}$$

This suggests us to define two real functions:

$$\begin{aligned} Y_1(x) &= \frac{1}{2}(y_1(x) + y_2(x)) = e^{Ax}\cos(Bx) \\ Y_2(x) &= \frac{1}{2i}(y_1(x) - y_2(x)) = e^{Ax}\sin(Bx) \end{aligned}$$

and hence the final solution is

$$\boxed{y_h(x) = C_1e^{Ax}\cos(Bx) + C_2e^{Ax}\sin(Bx)}$$

[Case 2, $p^2 - 4q = 0$**]:** In this case, we have repeated solution:

$$\lambda_{1,2} = \frac{-p}{2} \quad \Rightarrow \quad y_1(x) = e^{-\frac{p}{2}x} := e^{\lambda x}$$

For the second solution, we use variation of parameters:

$$y_2(x) = C(x)e^{\lambda x}$$

Substituting it into the DE, we obtain

$$y_2'' = C''e^{\lambda x} + \underbrace{\lambda C'e^{\lambda x} + \lambda C'e^{\lambda x}}_{=2\lambda C'e^{\lambda x}} + \lambda^2 Ce^{\lambda x}$$

Plug it into the DE:

$$\begin{aligned} \left(C''e^{\lambda x} + 2\lambda C'e^{\lambda x} + \lambda^2 Ce^{\lambda x}\right) + p\left(C'e^{\lambda x} + \lambda Ce^{\lambda x}\right) + qCe^{\lambda x} &= 0 \\ C''e^{\lambda x} + \underbrace{(2\lambda + p)}_{=0}C'e^{\lambda x} + \underbrace{(\lambda^2 + p\lambda + q)}_{=0}Ce^{\lambda x} &= 0 \\ C''(x) &= 0 \quad \Rightarrow \quad C(x) = x \end{aligned}$$

(Note that the constant term is absorbed in $y_1(x)$). This tells us that the general solution to the homogeneous DE is

$$\boxed{y_h(x) = (C_1 + C_2x)e^{\lambda x}, \qquad \lambda = \frac{-p}{2}}$$

Example 3.8. Solve the DE $y'' - y' - 2y(x) = 0$.

Solution. The characteristic equation is $\lambda^2 - \lambda - 2 = 0$, which has two distinct real solutions $\lambda_1 = 2$ and $\lambda_2 = -1$. Hence

$$y_h(x) = C_1e^{2x} + C_2e^{-x}$$

as desired. $\square$

Example 3.9. Solve the DE $y'' + 4y' + 5y(x) = 0$.

Solution. The characteristic equation is $\lambda^2 + 4\lambda + 5 = 0$, which has two distinct imaginary solutions $\lambda_1 = -2 - 2i$ and $\lambda_2 = -2 + 2i$. Hence

$$y_h(x) = C_1 e^{-2x} \cos(2x) + C_2 e^{-2x} \sin(2x)$$

as desired. $\square$

Example 3.10. Solve the DE $x'' + 6x' + 9x(t) = 0$.

Solution. The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$, which has a repeated real solution $\lambda_{1,2} = -3$. Hence

$$y_h(x) = C_1 e^{-3x} + C_2 x e^{-3x}$$

as desired. $\square$

3.4.2 Inhomogenous Linear 2nd Order Differential Equation with Constant Coefficients

Inhomogenous linear 2nd order differential equations with constant coefficients are of the form

$$A_2 y''(x) + A_1 y'(x) + A_0 y(x) = f(x)$$

In stardard form,

$$y'' + py' + qy = g$$

1. Find $y_h(x)$ using characteristic equation
2. Find a particular solution. If g is one of the special functions (polynomial, exponential, sine or cosine functions, linear combination of these or product of these functions), we use undetermined coefficients to obtain solution $y_p(x)$,
3. The final general solution is
$$y(x) = y_p(x) + y_h(x)$$

Example 3.11. Solve $y'' + y' - 6y(x) = e^{4x}$.

Solution. The homogeneous solution is

$$y_h(x) = C_1 e^{3x} + C_2 e^{-2x}$$

Our trial solution would be $y_p(x) = Ae^{4x}$ because function e^{4x} is exponential. Solving for A we get $A = \frac{1}{14}$, so our final solution is

$$y(x) = C_1 e^{3x} + C_2 e^{-2x} + \frac{1}{14} e^{4x}$$

as desired. $\square$

Example 3.12. Sovlve $y'' + y' - 6y(x) = x$.

Solution. The homogeneous solution is

$$y_h(x) = C_1 e^{3x} + C_2 e^{-2x}$$

Our trial solution would be $y_p(x) = Ax + B$. Solving for A and B we get $A = -1/6$ and $B = -1/36$. Hence our final general solution to the DE is

$$y(x) = C_1 e^{3x} + C_2 e^{-2x} - \frac{1}{6}x - \frac{1}{36}$$

as desired. $\square$

Example 3.13. Sovlve $y'' - 2y' + y(x) = \cos(2x)$.

Solution. The homogeneous solution is

$$y_h(x) = C_1 e^{x} + C_2 x e^{x}$$

Our trial solution would be $y_p(x) = A\cos(2x) + B\sin(2x)$. Solving for A and B we get $A = -3/25$ and $B = -4/25$. Hence our final general solution to the DE is

$$y(x) = C_1 e^{x} + C_2 x e^{x} - \frac{3}{25}\cos(2x) - \frac{4}{25}\sin(2x)$$

as desired. $\square$

Example 3.14. Solve the DE $y'' - y' - 2y = e^{2x}$.

Solution. The homogeneous solution is

$$y_h(x) = C_1 e^{2x} + C_2 e^{-x}$$

Our trial solution would be $y_p(x) = Ae^{2x}$, which is included in the homogeneous solution, so the actual trial solution is $y_p(x) = Axe^{2x}$. Solving for A we get $A = 1/3$ Hence our final general solution to the DE is

$$y(x) = C_1 e^{2x} + C_2 e^{-x} + \frac{1}{3}xe^{2x}$$

as desired. $\square$

3.5 Application

3.5.1 Undamped System

Simple Pendulum

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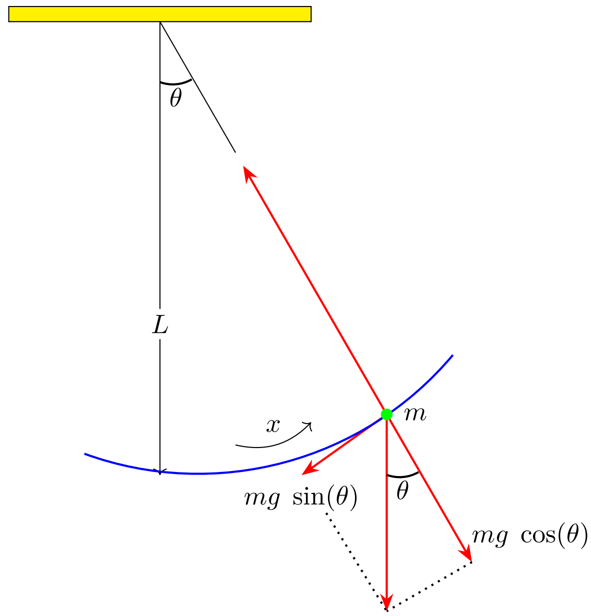
Suppose we have a simle pendulum:



We know that the only force experienced by tha object is

$$F = ma = -mg\sin\theta$$

where $a = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}$. We also know that for small angle θ, we have $s = L\theta$, so we obtain

$$a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = L \cdot \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

In particular, we have

$$mL\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -mg\sin\theta$$
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\sin\theta = 0$$

For small angles of θ, we have $\sin\theta \approx \theta$, so we have

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\theta = 0$$

define w_0 such that ${w_0}^2 = g/L$, so we have the second order differential equation:

$$\boxed{\theta'' + {w_0}^2\theta = 0}$$

Spring-mass Oscillation Let $x = 0$ be the equalibrium position. The force on the object is

$$F = ma = m\frac{\mathrm{d}^2x}{\mathrm{d}x^2} = -kx$$

so define w_0 be such that $w_0 = \sqrt{\frac{k}{m}}$, we have

$$\boxed{x'' + {w_0}^2 x = 0}$$

Let us try to solve this DE with constant coefficient. The characteristic equation is

$$\lambda^2 + {w_0}^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i w_0$$

so $x(t) = C_1 \cos(w_0 t) + C_2 \sin(w_0 t)$. We can write this solution in the form of

$$x(t) = R\sin(w_0 t - \phi) \quad \text{or} \quad x(t) = R\cos(w_0 t - \phi)$$

we can further write them as

$$x(t) = R\sin(w_0(t - \frac{\psi}{w_0}))$$

where R is the amplitude, ϕ is the phase angle, and ϕ/w_0 is the phase shift.

Example 3.15. Consider $x(t) = \sqrt{3}\cos(4t) + 1\sin(4t)$, write $x(t)$ in form of $x(t) = R\cos(4t - \phi)$ and determine R and ϕ.

Solution. We want $x(t) = R\cos(4t - \phi)$, using the formula of sum/difference of angles, we have

$$x(t) = R\cos(4t)\cos\phi + R\sin(4t)\sin\phi$$

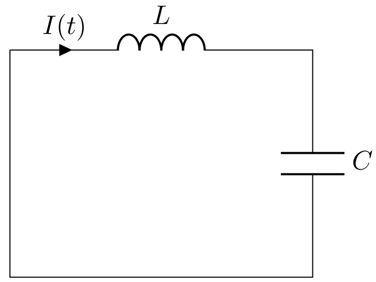
so we want

$$\begin{cases} R\cos\phi = \sqrt{3} \\ R\sin\phi = 1 \end{cases}$$

Solving the above system of equations we have $R = 2$ and $\phi = \pi/6$. □

Exercise 3.1. LC Circuit

Consider the LC circuit connected to voltage V_0.



For capacitor, the voltage drop is $V_C = \dfrac{Q(t)}{C}$, i.e., the drop is proportional to the charge $Q(t)$ (measured in Coulomb) and inversely proportional to the constant capacity C. For inductor, the voltage drop is $V_L(t) = L\dfrac{\mathrm{d}I(t)}{\mathrm{d}t}$, i.e., it is proportional to inductance L (measured in Henry) and proportional to the rate of change of current I. Note that $I(t) = \dfrac{\mathrm{d}Q}{\mathrm{d}t}$. Show that the model for LC circuit can be written as

$$I''(t) + {w_0}^2 I(t) = 0, \qquad w_0 = \frac{1}{LC}$$

Hint: Kirchkoff's voltage law: In a closed loop in an electric circuit, we have the following

$$\sum V_{\text{source}} = \sum V_{\text{drops}}$$

3.5.2 Dampled System

Spring-mass Oscillation We assume that we have a damper which is directly proportional to the velocity of the mass and opposes the movement. In this case, we have

$$mx'' = -kx(t) - cx' \quad \Rightarrow \quad x''' + \frac{c}{m}x' + \frac{k}{m}x(t) = 0$$

Again define w_0 be such that ${w_0}^2 = \frac{k}{m}$. The damping parameter is $\zeta = \frac{c}{2w_0 n}$, which is a positive dimensionless constant. Hence the equation becomes

$$\boxed{x'' + 2w_0\zeta x' + {w_0}^2 x = 0}$$

Definition 3.5. **[Standard Form of Damped Oscillation System]**

The boxed equation right above is the standard form of a damped oscillation system.

Note 3.2. Similar DE can be written for other physical systems such as damped pendulum and RLC circuit.

Again, let's try to solve this constant coefficient DE: The characteristic equation is

$$\lambda^2 + 2w_0\zeta\lambda + {w_0}^2 = 0$$

which has two roots

$$\lambda_{1,2} = \frac{-2w_0\zeta \pm \sqrt{(2w_0\zeta)^2 - 4{w_0}^2}}{2} = (-\zeta \pm \sqrt{\zeta^2 - 1})w_0$$

Depending on the value of the discriminant, $\zeta^2 - 1$, we have three difference cases.

($\zeta^2 - 1 > 0$). This is called a **overdamped system**. In this case, $\zeta \in (1, \infty)$, and that we have two real roots λ_1 and λ_2. The solution is

$$\begin{aligned} x(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ &= C_1 e^{(-\zeta + \sqrt{\zeta^2-1})w_0 t} + C_2 e^{(-\zeta - \sqrt{\zeta^2-1})w_0 t} \end{aligned}$$

Notice that as $t \to \infty$, we have $x \to 0$.

> **Comment 3.2.** There is no oscillation for overdamped systems.

($\zeta^2 - 1 < 0$). In this case, $\zeta \in (0, 1)$. Note that if $\zeta = 0$ then this is a **underdamped system**. We have two complex conjugate roots,

$$\lambda_{1,2} = -w_0\zeta \pm i w_0 \sqrt{1 - \zeta^2}$$

Hence the solution is

$$x(t) = e^{-w_0\zeta t}\left[C_1 \cos(w_0\sqrt{1-\zeta^2}t) + C_2 \sin(w_0\sqrt{1-\zeta^2}t)\right]$$

or

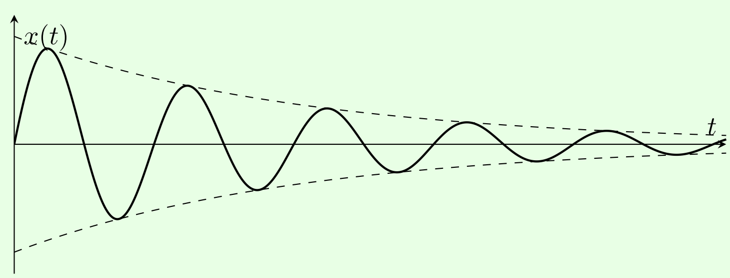
$$x(t) = Re^{-w_0\zeta t}\cos(w_0\sqrt{1-\zeta^2}t - \phi)$$

$Re^{-w_0\zeta t}$ is called a decaying amplitude because it decreases as t increases. Since $-1 \le \cos(\bullet) \le 1$, so

$$-Re^{-w_0\zeta t} \le x(t) \le Re^{-w_0\zeta t} \quad \Rightarrow \quad |x(t)| \le Re^{-w_0\zeta t}$$

so again, $x(t) \to 0$ as $t \to \infty$.

Example 3.16.



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($\zeta^2 - 1 = 0$). This is called a **critically damped system**. In this case, $\zeta = 1$ is a repeated root for the characteristic equation:

$$\lambda_1 = \lambda_2 = -w_0\zeta = -w_0$$

Hence

$$x(t) = C_1 e^{-w_0 t} + C_2 t e^{-w_0 t}$$

The solution is similar to overdamped (no oscillation) and $x(t) \to 0$ faster than in the overdamped system.

Discovery 3.1. Critically damped system vanishes much faster than overdamped system.

Example 3.17. A mechanical ststem undergoes damped oscillations governed by the DE

$$y'' + 2by' + {w_0}^2 y(t) = 0$$

Suppose that, over 10 oscillations, the amplitude of the oscillations drops from 27cm to 10cm. Find the value of dimensionless damping arameter $\zeta = \frac{b}{w_0}$.

Solution. Standard DE for damped oscillator is

$$y'' + 2\zeta w_0 y' + {w_0}^2 y(t) = 0$$

Comparing the given DE with the one above, we identify $\zeta = \frac{b}{w_0}$. At t_1, we have

$$R_1 = 27cm = Re^{-w_0 \zeta t_1}$$

At $t_2 = t_1 + 10\tilde{T}$, we have

$$R_2 = 10cm = Rw^{-w_0 \zeta t_2}$$

Now, we have

$$\frac{R_1}{R_2} = \frac{Re^{-w_0 \zeta t_1}}{Re^{-w_0 \zeta \left(t_1 + 10 \cdot \frac{2\pi}{w_0 \sqrt{1-\zeta^2}}\right)}} = \frac{27}{10} \quad \Rightarrow \quad e^{\frac{20\pi\zeta}{\sqrt{1-\zeta^2}}} = \frac{27}{10}$$

Solving for ζ, we will find $\zeta \approx 0.016$. □

Example 3.18. Decide which system describes overdamped, underdamped, or critically damped motion.

(a) $x'' + 2x' + 3x(t) = 0$;

(b) $x'' + \frac{7}{2}x' + \frac{3}{2}x(t) = 0$;

(c) $x'' + 4x' + 4x(t) = 0$;

(d) $x'' + \frac{1}{4}x(t) = 0$.

Solution. (a) Recall the standard form,

$$x'' + 2w_0 \zeta x' + {w_0}^2 x(t) = 0$$

We identify that $w_0 = \sqrt{3}$ and $\zeta = \frac{1}{\sqrt{3}} < 1$, so this is an underdamped system. Alternatively, we could also

find that the characteristic equation is
$$\lambda^2 + 2\lambda + 3 = 0$$
which has two imaginary roots, this again verifies that the system is underdamped.
(b) This is an overdamped system.
(c) This is a critically damped system.
(d) This is an undamped system, $\zeta = 0$. □

3.5.3 Damped Forced Oscillation System (Periodic Force)

The standard form is
$$x''(t) + 2\zeta w_0 x'(t) + {w_0}^2 x(t) = f_0 \cos(wt)$$
which is an inhomogeneous 2nd order linear DE with constant coefficients. We know that the general solution is of the form
$$x(t) = x_h(t) + x_p(t)$$
For homogeneous solution to the homogeneous equation
$$x''(t) + 2\zeta w_0 x'(t) + {w_0}^2 x(t) = 0$$
we would have one of the three cases introduced in the previous subsubsection, they are overdamped, underdamped, and critically damped. For all these cases, we know
$$x_h(t) \to 0 \qquad \text{as} \qquad t \to \infty$$
As a result, in the solution
$$x(t) = x_h(t) + x_p(t)$$
We call $x_h(t)$ the **transient part** of the solution, and on the opposite, $x_p(t)$ is called the **steady state part** of the solution.

Discovery 3.2. We can find $x_p(t)$ using undertermined coefficient.

Using undetermined coefficients, we trial
$$\begin{aligned}
x_p(t) &= A\cos(wt) + B\sin(wt) \\
x_p'(t) &= -Aw\sin(wt) + Bw\cos(wt) \\
x_p''(t) &= -Aw^2\cos(wt) - Bw^2\sin(wt)
\end{aligned}$$
Substituting these into the DE, we obtain
$$\begin{aligned}
\left[-Aw^2\cos(wt) - Bw^2\sin(wt)\right] &+ 2\zeta w_0\left[-Aw\sin(wt) + Bw\cos(wt)\right] \\
&+ {w_0}^2\left[A\cos(wt) + B\sin(wt)\right] = f_0\cos(wt)
\end{aligned}$$

Solving for A and B we obtain

$$A = \frac{f_0({w_0}^2 - w^2)}{({w_0}^2 - w^2)^2 + (2\zeta w_0 w)^2} \qquad\qquad B = \frac{f_0(2\zeta w_0 w)}{({w_0}^2 - w^2)^2 + (2\zeta w_0 w)^2}$$

We can write $x_p(t) = A_{SS}\cos(wt - \delta)$, where A_{SS} is the steady state amplitude. Expand $\cos(wt - \delta)$, we have

$$x_p(t) = \underbrace{A_{SS}\cos\delta}_{=A}\cos(wt) + \underbrace{A_{SS}\sin\delta}_{=B}\sin(wt)$$

Hence we could figure out that $A_{SS} = \sqrt{A^2 + B^2}$ and hence

$$A_{SS}^2 = \frac{{f_0}^2}{({w_0}^2 - w^2)^2 + (2\zeta w_0 w)^2} \quad \Rightarrow \quad A_{SS} = \frac{f_0}{\sqrt{({w_0}^2 - w^2)^2 + (2\zeta w_0 w)^2}}$$

Note 3.3. Note that this tells us that

$$\sin\delta = \frac{B}{A_{SS}} > 0$$

which implies that $\delta \in (0, \pi)$.

Therefore, we have

$$x(t) = x_h(t) + \frac{f_0}{\sqrt{({w_0}^2 - w^2)^2 + (2\zeta w_0 w)^2}}\cos(wt - \delta), \quad \delta \in (0, \pi)$$

Analysis of Steady State Amplitude: Suppose $w = 0$, then

$$A_{SS}(w = 0) = \frac{f_0}{{w_0}^2}$$

Define

$$\mathcal{A} = \frac{A_{SS}}{f_0/{w_0}^2} \qquad \text{and} \qquad \Omega = \frac{w}{w_0}$$

we know that they are dimensionless. In particular, we calculate

$$\mathcal{A} = \frac{{w_0}^2}{f_0}\frac{f_0}{\sqrt{({w_0}^2 - w^2)^2 + (2\zeta w_0 w)^2}} = \frac{{w_0}^2}{{w_0}^2\sqrt{\left(1 - \frac{w^2}{{w_0}^2}\right)^2 + \left(2\zeta\frac{w}{w_0}\right)^2}}$$

This implies that

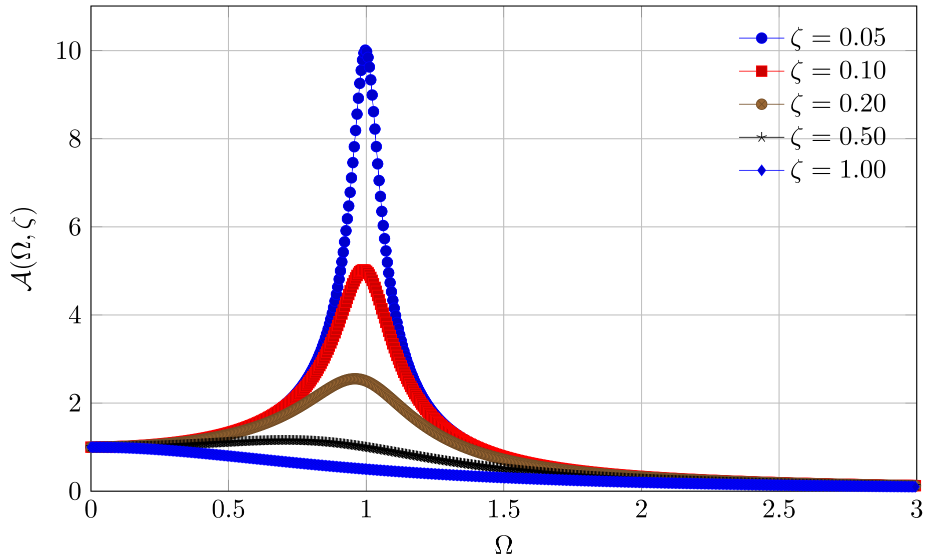
$$\mathcal{A}(\Omega, \zeta) = \frac{1}{\sqrt{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}}$$

Constructing the graph of $\mathcal{A}$ as a function of Ω for fixed ζ, then change the values of ζ:

(1) $\mathcal{A}(0, \zeta) = 1$ for all $\zeta \geq 0$;

(2) $\lim \mathcal{A}(\Omega, \zeta) = 0$ as $\Omega \to \infty$;

(3) $\mathcal{A}(\Omega, \zeta) = \frac{1}{\sqrt{1 + \Omega^4 + 2\Omega^2(2\zeta^2 - 1)}} < 1$ if $2\zeta^2 \geq 1$.



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If $\zeta < \frac{1}{\sqrt{2}}$, then

$$\mathcal{A}(\Omega,\zeta) = \frac{1}{\sqrt{1+\Omega^4-2\Omega^2+4\zeta^2\Omega^2}}$$

We notice that $\mathcal{A}(\Omega,\zeta) = 1$ when $\Omega = 0$. At $\Omega^2 - 2(1-2\zeta^2) = 0$, we have

$$\Omega = \sqrt{2(1-2\zeta^2)}$$

This means that we have a maximum for $\mathcal{A}(\Omega,\zeta)$ for $\zeta < \frac{1}{\sqrt{2}}$. To find the maximum, we need to compute $\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Omega}$:

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Omega} = -\frac{1}{2}\left[1+\Omega^4-2\Omega^2+4\zeta^2\Omega^2\right]^{-3/2}\left[4\Omega^3-4\omega(1-2\zeta^2)\right]$$

Solving for $\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Omega} = 0$, we have

$$\Omega = 0 \qquad \text{or} \qquad \Omega = \sqrt{1-2\zeta^2}$$

Substituting in the latter, we obtain

$$\mathcal{A}(\sqrt{1-2\zeta^2},\zeta) = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

When the damping parameter satisfies $0 < 2\zeta^2 < 1$, then $\mathcal{A}$ attains a maximum value greater than 1. We say that the system undergoes **resonance**. We had $\Omega = \frac{w}{w_0}$, $\mathcal{A} = A_{SS}/(f_0/{w_0}^2)$, so

$$\Omega_{\text{res}} = \sqrt{1-2\zeta^2}, \qquad \mathcal{A}_{\text{res}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

This tells us that

$$\boxed{w_{\text{res}} = w_0\sqrt{1-2\zeta^2}, \qquad A_{SS\text{res}} = \frac{f_0}{2{w_0}^2\zeta\sqrt{1-\zeta^2}}}$$

3.5.4 Undamped Forxed Oscillation (Beats)

Recall that undamped means that $\zeta = 0$. The differential equation in this scenario is

$$x''(t) + {w_0}^2 x(t) = f_0 \cos(wt)$$

which has the solution of the form

$$x(t) = x_h(t) + x_p(t)$$

For the homogeneous solution, we have the characteristic equation

$$\lambda^2 {w_0}^2 = 0 \qquad \Rightarrow \qquad \lambda_{1,2} = \pm i w_0$$

and hence

$$x_h(t) = C_1 \cos(w_0 t) + C_2 \sin(w_0 t)$$

Assuming that $w \neq w_0$,

$$x_p(t) = A\cos(wt) + B\sin(wt)$$

We can substitute into the DE and find A and B, or let $\zeta = 0$ in $x_p(t)$ we found for damped force system:

$$x_p = \frac{f_0}{\sqrt{({w_0}^2 - w^2)^2 + (2\zeta w_0 w)^2}} \cos(wt - \phi)$$

Letting $\phi = 0$, we have

$$x_p(t) = \frac{f_0}{{w_0}^2 - w^2} \cos(wt - \phi)$$

so

$$\cos\phi = \frac{{w_0}^2 - w^2}{\sqrt{({w_0}^2 - w^2)^2 + (2\zeta w_0 w)^2}} = 1, \qquad \sin\phi = \frac{2\zeta w_0 w}{\sqrt{({w_0}^2 - w^2)^2 + (2\zeta w_0 w)^2}} = 0$$

and thus $\phi = 0$. This further implies that

$$x_p(t) = \frac{f_0}{{w_0}^2 - w^2} \cos(wt)$$

Assuming now we have initial conditions $x(0) = x'(0) = 0$. Subsituting in, we can find that

$$C_1 = -\frac{f_0}{{w_0}^2 - w^2}, \qquad C_2 = 0$$

Now, we have

$$\begin{aligned} x(t) &= -\frac{f_0}{{w_0}^2 - w^2} \cos(w_0 t) + \frac{f_0}{{w_0}^2 - w^2} \cos(wt) \\ &= \frac{f_0}{{w_0}^2 - w^2} [\cos(wt) - \cos(w_0 t)] \\ &= \boxed{\frac{-2f_0}{{w_0}^2 - w^2} \sin\left(\frac{w - w_0}{2} t\right) \sin\left(\frac{w + w_0}{2} t\right)} \end{aligned}$$

Let's now compare these sine functions:

$$\sin\left(\frac{w-w_0}{2}t\right): \qquad T = \frac{2\pi}{|w-w_0|/2} = \frac{4\pi}{|w-w_0|}$$
$$\sin\left(\frac{w+w_0}{2}t\right): \qquad T = \frac{4\pi}{w+w_0}$$

Note 3.4. If $w \approx w_0$, then $T_1 \ll T_2$.

3.5.5 Undamped Forced Oscillation (Extreme Form of Resonance if $w = w_0$)

In this case, we have

$$x''(t) + {w_0}^2 x(t) = f_0 \cos(w_0 t)$$

We solve this either by finding homogeneous solution

$$x_h(t) = C_1 \cos(w_0 t) + C_2 \sin(w_0 t)$$

and then define $x_p(t) = At\cos(w_0 t) + Bt\sin(w_0 t)$. Substituting x_p into the DE and find A and B. Or we could use the solution for undamped forced oscillations and find the limit

$$\lim_{w \to w_0} x(t)$$

For initial conditions $x(0) = x'(0) = 0$, recall

$$C_1 = \frac{-f_0}{{w_0}^2 - w^2}, \qquad C_2 = 0$$

and

$$x(t) = \frac{f_0}{{w_0}^2 - 2^2}[\cos(wt) - \cos(w_0 t)]$$

Using L'Hopital's rule, we can find that

$$\lim_{w \to w_0} x(t) = \frac{f_0}{2w_0} t \sin(w_0 t)$$

3.6 Cauchy-Euler Equation

What if our DE is not a constant coefficient DE? Suppose our DEs are of the form

$$x^2 y''(x) + pxy'(x) + qy(x) = 0$$

where p and q are constants.

1. We use transformation $x = e^u$ or $u = \ln x$ to transform the DE to a constnat coefficient DE.

$$y' = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\left(\frac{1}{x}\right)$$

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}u}\left(\frac{1}{x}\right)\right) = \frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - \frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}u}$$

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2. Substituting into the DE:

$$x^2\left[\frac{1}{x^2}\frac{d^2y}{du^2} - \frac{1}{x^2}\frac{dy}{du}\right] + px\left[\frac{dy}{du}\left(\frac{1}{x}\right)\right] + qy = 0,$$

$$\frac{d^2y}{du^2} - \frac{dy}{du} + p\frac{dy}{du} + q\,y(u) = 0,$$

$$\boxed{\frac{d^2y}{du^2} + (p-1)\frac{dy}{du} + q\,y(u) = 0}$$

which is a constant coefficient DE.

3. Solve this DE for $y(u)$,

4. Return to our original variable $u = \ln x$.

Example 3.19. Find the solution to the DE:

$$x^2y'' + xy' - y(x) = 0$$

Solution. This is a Cauchy-Euler equation, set $u = \ln x$. We identify that $p = 1$ and $q = -1$, so

$$y'' + (1-1)y' - y(u) = 0$$

Solving for this we know

$$y(u) = C_1e^u + C_2e^{-u}$$

so the general solution is

$$y(x) = C_1x + C_2\frac{1}{x}$$

□

3.6.1 Method of Reduction of Order if one solution to homogeneous DE is given

Assume we have DE of then form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = F(x)$$

If $y_1(x)$ is a non-trivial solution of the homogeneous DE, then the general solution to the DE is of the form

$$\boxed{y(x) = v(x)y_1(x)}$$

1. Standard form: $y''(x) + p(x)y'(x) + q(x)y(x) = g(x)$
2. For $y_1(x)$, we know that $y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = g(x)$
3. Define $y(x) = v(x)y_1(x)$ and substitute it into the DE, we first note that

$$\begin{aligned} y' &= v'y_1 + vy_1' \\ y'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

   Hence

$$v''y_1 + (2y_1' + p(x)y_1)v' + \underbrace{(y_1'' + p(x)y_1' + q(x)y)}_{=0}v = g(x)$$

   where the dependent variable is mising, so we perform reduction of order: define $v' = w$ and so $v'' = w'$,

$$w'y_1 + (2y_1' + p(x)y_1)w = g(x)$$

   Solve the DE for $w(x)$.
4. $v' = w(x)$, so $v(x) = \int w(x)\,\mathrm{d}x + C$,
5. The general solution to the DE is $y(x) = v(x)y_1(x)$.

Example 3.20. Solve the DE:

$$x^2y'' + xy' - y(x) = 0$$

where $y_1(x) = x$.

Solution. Since one solution is already known, we use *reduction of order*. Let

$$y(x) = v(x)y_1(x) = x\,v(x).$$

Then we know that

$$\begin{aligned} y' &= v + xv', \\ y'' &= v' + (v' + xv'') = 2v' + xv''. \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} x^2(2v' + xv'') + x(v + xv') - xv &= 0, \\ 2x^2v' + x^3v'' + xv + x^2v' - xv &= 0, \\ x^3v'' + 3x^2v' &= 0. \end{aligned}$$

For $x \neq 0$, divide by x^2:

$$xv'' + 3v' = 0.$$

Since the dependent variable v is missing, set $w = v'$. Then the equation becomes

$$xw' + 3w = 0,$$

which is separable. Hence

$$\begin{aligned} \frac{dw}{w} &= -\frac{3}{x}\,dx, \\ \ln|w| &= -3\ln|x| + C, \\ \ln|wx^3| &= C. \end{aligned}$$

Therefore,

$$wx^3 = C_1 \qquad \Rightarrow \qquad w = \frac{C_1}{x^3}.$$

So

$$v'(x) = \frac{C_1}{x^3} = C_1 x^{-3}.$$

Integrating,

$$\begin{aligned} v(x) &= C_1 \int x^{-3}\,dx + C_2 \\ &= C_1 \left(\frac{x^{-2}}{-2} \right) + C_2 \\ &= -\frac{C_1}{2} x^{-2} + C_2. \end{aligned}$$

Rename the constant $-\frac{C_1}{2}$ as C_3. Then

$$v(x) = C_3 x^{-2} + C_2.$$

Finally,

$$\begin{aligned} y(x) &= v(x)\,y_1(x) \\ &= \left(C_3 x^{-2} + C_2 \right) x \\ &= \frac{C_3}{x} + C_2 x. \end{aligned}$$

Thus the general solution is

$$\boxed{y(x) = C_2 x + \frac{C_3}{x}}.$$

□

Example 3.21. Solve the IVP:

$$x^2y'' + xy' + 4y(x) = 5(x + x^3), \qquad y(1) = 0, y'(1) = 0$$

Solution. This is a second-order linear differential equation with variable coefficients, so we write

$$y(x) = y_h(x) + y_p(x).$$

First, find the homogeneous solution $y_h(x)$ from

$$x^2y'' + xy' + 4y(x) = 0.$$

This is a Cauchy-Euler equation with $p = 1$ and $q = 4$. Let

$$x = e^u, \qquad u = \ln x.$$

Then the equation becomes

$$y''(u) + (p - 1)y'(u) + q\,y(u) = 0.$$

Substituting $p = 1$ and $q = 4$ gives

$$y''(u) + (1 - 1)y'(u) + 4y(u) = 0 \quad \Longrightarrow \quad y''(u) + 4y(u) = 0.$$

Hence

$$y_h(u) = c_1 \cos(2u) + c_2 \sin(2u).$$
$$y_h(x) = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x).$$

For a particular solution, try

$$y_p(x) = Ax^3 + Bx^2 + Cx + D.$$

Then

$$y_p'(x) = 3Ax^2 + 2Bx + C,$$
$$y_p''(x) = 6Ax + 2B.$$

Substitute into the DE, we get

$$x^2(6Ax + 2B) + x(3Ax^2 + 2Bx + C) + 4(Ax^3 + Bx^2 + Cx + D) = 5(x + x^3).$$

The rest are easy.

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An alternative solution is to work with transformed DE:

$$y'' + 4y = 5x + 5x^3 = 5e^u + 5e^{3u}$$

where $x = e^u$. We are working with $y'' + 4y = 5e^u + 5e^{3u}$. For homogeneous solution,

$$y_h(x) = C_1 \cos(2u) + C_2 \sin(2u)$$

We then trial $y_p(u) = Ae^u + Be^{3u}$, substituting into the transformed DE, we solve for A and B and obtain that $A = 1$, $B = 5/13$. We then apply the initial conditions and get that $C_1 = -18/13$, $C_2 = -28/26$ $\square$

Exercise 3.2. Try the transformation $y(x) = x^r$ for a constant r in the previous example to find the homogeneous solution, and then use the undertermined coefficient method to find a particular solution.

3.6.2 Variation of Parameters

We start with the general form:

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$$

Here is our procedure:

1. Rewrite the DE in the standard form:

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x)$$

2. Solve the homogeneous DE $y''(x) + p(x)y'(x) + q(x)y(x) = 0$, say the solution $y_h(x) = C_1 y_1(x) + C_2 y_2(x)$.

3. Consider the solution
$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

4. Substituting the solution $y_p(x)$ into the DE, the resulting DE can be rewritten as

$$\begin{aligned} u_1 \underbrace{(y_1'' + p(x)y_1' + q(x)y_1)}_{=0} + u_2 \underbrace{(y_2'' + p(x)y_2' + q(x)y_2)}_{=0} \\ + u_1''y_1 + 2u_1'y_1' + u_2''y_2 + 2u_2'y_2' + p(x)(u_1'y_1 + u_2'y_2) = g(x) \end{aligned}$$

   If we assume that $\boxed{u_1'y_1 + u_2'y_2 = 0}$, then

$$u_1''y_1 + u_1'y_1' + u_2''y_2 + u_2'y_2' = (u_1'y_1 + u_2'y_2)' = 0$$

   so we have
$$\boxed{u_1'y_1' + u_2'y_2' = g(x)}$$

   and solve for u_1 and u_2.

Example 3.22. Solve the DE:

$$y'' + 4y(x) = \sec(2x), \qquad \frac{-\pi}{4} < x < \frac{\pi}{4}$$

Solution. We easily find the homogeneous solution:

$$y_h(x) = C_1 \cos(2x) + C_2 \sin(2x)$$

Hence we trial

$$y_p(x) = u_1(x)\cos(2x) + u_2(x)\sin(2x)$$

We try to solve the system of equations:

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g(x) \end{cases} \qquad \Rightarrow \qquad \begin{cases} u_1'\cos(2x) + u_2'\sin(2x) = 0 \\ -2u_1'\sin(2x) + 2u_2'\cos(2x) = \sec(2x) \end{cases}$$

Solving for u_1 and u_2 we get $u_2 = \frac{x}{2}$ and $u_1 = \frac{1}{4}\ln(\cos(2x))$. □

3.6.3 A note for Boundary Value Problems (BVPs)

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x), \qquad y(x_0) = y_0, y(x_1) = y_1$$

Discovery 3.3. A BVP may have no solution, one solution, or so many solutions.

Example 3.23. Solve the DE subject to the BCs:

$$y'' + \pi^2 y(x) = 0$$

(a) y(0) = 0, $y(1) = 1$;

(b) y(0) = 0, $y(1) = 0$.

Solution. We find that the general solution is

$$y(x) = C_1\cos(\pi x) + C_2\sin(\pi x)$$

For (a), we will find that there is no solution. For (b), we will find that C_2 is a free parameter. □

4 Introduction to Laplace Transform (LT)

4.1 Definitions and Examples

Definition 4.1. **[Laplace Transform]**

If $f(t)$ is defined for $t \geq 0$, then the Laplace transform $L\{f(t)\}$ is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

for all values of s such that the improper integral converges.

Example 4.1. Find LT of $f(t) = c$ where c is a constnat.

Solution. We have

$$\begin{aligned} F(s) = L\{c\} &= \int_0^\infty e^{-st} c\,\mathrm{d}t \\ &= c \lim_{u\to\infty} \int_0^u e^{-st}\,\mathrm{d}t \\ &= c \lim_{u\to\infty} \left[-\frac{1}{s} e^{-st}\right]_0^u \\ &= c\left[\frac{1}{s} + \lim_{u\to\infty} -\frac{1}{s} e^{-st}\right] \end{aligned}$$

which equals to c/s if $s > 0$. ☐

Example 4.2. Find $L\{e^{\alpha t}\}$ if α is a constant.

Solution. We have

$$\begin{aligned} F(s) = L\{e^{\alpha t}\} &= \int_0^\infty e^{-st} e^{\alpha t}\,\mathrm{d}t \\ &= \lim_{u\to\infty} \int_0^u e^{-(s-\alpha)t}\,\mathrm{d}t \\ &= \frac{1}{-(s-\alpha)} \lim_{u\to\infty} \left[e^{-(s-\alpha)t}\right]_0^u \\ &= \frac{-1}{s-\alpha}\left[1 - \lim_{u\to\infty} e^{-(s-\alpha)u}\right] \end{aligned}$$

which equals to $1/(s-\alpha)$ if $s > \alpha$. ☐

Exercise 4.1. Show that $L\{t^n\} = \frac{n!}{s^{n+1}}$ for $s > 0$.
Hint: Show that $L\{t\} = \frac{1}{s^2}$ *and* $L\{t^2\} = \frac{2}{s^3}$, *this suggests using induction.*

4.2 Properties of Laplace Transform

Proposition 4.1. linearity of Laplace Transform

Suppose $\mathcal{L}\{f(t)\} = F(s), \mathcal{L}\{g(t)\} = G(s)$, then we have

$$\begin{aligned}\mathcal{L}\{c_1 f(t) + c_2 g(t)\} &= c_1\,\mathcal{L}\{f(t)\} + c_2\,\mathcal{L}\{g(t)\}\\ &= c_1 F(s) + c_2 G(s).\end{aligned}$$

Example 4.3. Can you find $\mathcal{L}\{\sin(bt)\},\ \mathcal{L}\{\cos(bt)\}$?

Solution. We know that $e^{ibt} = \cos(bt) + i\sin(bt)$, so

$$\mathcal{L}\{e^{ibt}\} = \mathcal{L}\{\cos(bt)\} + i\,\mathcal{L}\{\sin(bt)\}$$

Moreover, we know that

$$\frac{1}{s-ib} = \mathcal{L}\{\cos(bt)\} + i\,\mathcal{L}\{\sin(bt)\}$$

and

$$\frac{1}{s-ib}\cdot\frac{s+ib}{s+ib} = \frac{s}{s^2+b^2} + i\,\frac{b}{s^2+b^2}$$

Hence $\mathcal{L}\{\cos(bt)\} = \dfrac{s}{s^2+b^2}, \quad \mathcal{L}\{\sin(bt)\} = \dfrac{b}{s^2+b^2}.$ □

4.2.1 Shift Theorems

Theorem 4.1. First Shift Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a > 0$, then $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$ for $s > c + a$.

Example 4.4. Find $\mathcal{L}\{e^{3t}t^2\}$.

Solution. We know that

$$\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

Hence we know that

$$\mathcal{L}\{e^{3t}t^2\} = \frac{2}{(s-3)^3}, \qquad s > 3$$

□

Theorem 4.2. Second Shift Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a > 0$ and c is a positive constant, then $\mathcal{L}\{H(t-c)f(t-c)\} = e^{-sc}F(s)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 1 \end{cases}$.

Example 4.5. Let's define $f(t-c)=1$, and we know

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s>0$$

From the above theorem, we know that

$$\mathcal{L}\{H(t-c)1\} = e^{-sc}\mathcal{L}\{1\} = \frac{1}{s}e^{-sc}$$

Taking $c=0$ we get $\mathcal{L}\{H(t)\} = \frac{1}{s}$.

Theorem 4.3. AlternativeSecond Shift Theorem

We have

$$\mathcal{L}\{H(t-c)f(t)\} = e^{-sc}\mathcal{L}\{f(t+c)\}, \qquad \text{for } s>a$$

Exercise 4.2. Prove the theorem using the integral definition of LT (Hint: take $u=t-c$, or $t=u+c$).

Example 4.6. Find $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} t & t\in[0,2) \\ 0 & t\in[2,\infty) \end{cases}$.

Solution. We notice that

$$f(t) = t[H(t-0)-H(t-2)]$$

Hence

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{tH(t)\} - \mathcal{L}\{tH(t-2)\} \\ &= \mathcal{L}\{tH(t)\} - \mathcal{L}\{tH(t-2)+2H(t-2)-2H(t-2)\} \\ &= \mathcal{L}\{tH(t)\} - \mathcal{L}\{(t-2)H(t-2)\} + 2\mathcal{L}\{H(t-2)\}\end{aligned}$$

Recall the Second Shift Theorem, $\mathcal{L}\{H(t-c)f(t-c)\} = e^{-cs}F(s)$, so

$$\begin{aligned}\mathcal{L}\{f(t)\} &= e^{-s(0)}\mathcal{L}\{t\} - e^{-s(2)}\mathcal{L}\{t\} + 2e^{-s(2)}\mathcal{L}\{1\} \\ &= \frac{1}{s^2} - e^{-2s}\frac{1}{s^2} + 2e^{-2s}\frac{1}{s}\end{aligned}$$

as desired. □

Theorem 4.4. Derivative Theorem

If $F(s)=\mathcal{L}\{f(t)\}$ exists for $s>a$, then

1. $F^{(n)}(s) = (-1)^n\mathcal{L}\{t^n f(t)\}$ or;
2. $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) = (-1)^n\frac{\mathrm{d}^n F}{\mathrm{d}s^n}$

Exercise 4.3. Let

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Show that

$$\frac{dF}{ds} = -\mathcal{L}\{t\, f(t)\},$$

and, more generally, for any integer $n \geq 1$,

$$\frac{d^n F}{ds^n} = (-1)^n\, \mathcal{L}\{t^n f(t)\}.$$

Example 4.7. Find $\mathcal{L}\{t\cos(bt)\}$.

Solution. We know that

$$\mathcal{L}\{tf(t)\} = (-1)^1 \frac{\mathrm{d}F}{\mathrm{d}s} = -\frac{\mathrm{d}F}{\mathrm{d}s}$$

and we also know that

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$$

so we can get that

$$\mathcal{L}\{t\cos(bt)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+b^2}\right) = -\frac{b^2-s^2}{(s^2+b^2)^2}$$

as desired. □

4.2.2 Laplace Transform of Derivatives

Proposition 4.2. If $f(t) = O(e^{at})$ for some a and $f'(t)$ is a piecewise continuous function and $f(t)$ is continuous on $[0, r]$, then $\mathcal{L}\{f'(t)\}$ exists and is

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0), \qquad s > 0$$

Proposition 4.3. If $f(t) = O(e^{at})$ and $f'(t) = O(e^{at})$ for some a, and $f''(t)$ is a piecewise continuous function and both $f(t)$ and $f'(t)$ are continous on any interval $[0, r]$, then $\mathcal{L}\{f''(t)\}$ exists and is

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0), \qquad s > 0$$

Example 4.8. Find $\mathcal{L}\{y(t)\} = Y(s)$, where $y(t)$ is a solution to the following IVP:

$$y'(t) - y(t) = 2te^{-t}, \qquad y(0) = y_0$$

Solution. Apply LT to both sides of the DE:

$$\mathcal{L}\{y'(t)\} - \mathcal{L}\{y(t)\} = 2\mathcal{L}\{te^{-t}\}$$
$$(sY(s) - y(0)) - Y(s) = 2\mathcal{L}\{te^{-t}\}$$

We know that $\mathcal{L}\{t\} = 1/s^2$, so from the First Shift Theorem

$$\mathcal{L}\{e^{-tt}\} = F(s+1) = \frac{1}{(s+1)^2}$$

Hence

$$(sY(s) - y(0)) - Y(s) = \frac{2}{(s+1)^2}$$

This gives us that

$$Y(s) = \frac{2}{(s+1)^2(s-1)} + \frac{y_0}{(s-1)}$$

□

4.3 Inverse Laplace Transform

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We need to know if a given $F(s)$ will yield to a unique $f(t)$.

Proposition 4.4. If $f_1(t)$ and $f_2(t)$ are continuous functions of exponential order e^{at}, then

$$f_1(t) \neq f_2(t) \;\Rightarrow\; \mathcal{L}\{f_1(t)\} \neq \mathcal{L}\{f_2(t)\}.$$

Comment 4.1. As a result of the above proposition, the Laplace transform is a one-to-one operator and has an inverse operator $\mathcal{L}^{-1}$, which maps the Laplace transform $F(s)$ onto the original function $f(t)$.

Note 4.1. The L^{-1} (inverse Laplace transform) is also a linear operator:

$$\begin{aligned}\mathcal{L}^{-1}\{c_1F_1(s) + c_2F_2(s)\} &= \mathcal{L}^{-1}\{c_1F_1(s)\} + L^{-1}\{c_2F_2(s)\}\\ &= c_1f_1(t) + c_2f_2(t)\end{aligned}$$

Example 4.9. We compute the inverse Laplace transform of

$$\frac{1}{s^2+2s+10}.$$

Solution. First recall the basic Laplace transform formulas

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t, \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Also recall the shift property

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

We begin by completing the square in the denominator:

$$s^2 + 2s + 10 = (s+1)^2 + 9 = (s+1)^2 + 3^2.$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+3^2}\right\}.$$

To match a standard Laplace transform formula, rewrite the expression as

$$\frac{1}{(s+1)^2+3^2} = \frac{1}{3}\,\frac{3}{(s+1)^2+3^2}.$$

Recall the known transform

$$\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+3^2}.$$

Using the shift property,

$$\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+3^2}\right\} = e^{-t}\sin(3t).$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+10}\right\} = \frac{1}{3}e^{-t}\sin(3t).$$

□

Exercise 4.4. Show that

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+5s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+2)(s+3)}\right\}$$

4.4 Solving Initial Value Problems with Laplace Transform

1. Apply the Laplace transform to both sides of the differential equation, and use the table for Laplace transforms of functions and derivatives.
2. Substitute the initial conditions.
3. Rearrange the terms and solve for $Y(s)$.
4. Take the inverse Laplace transform to find $y(t)$, where $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Example 4.10. Solve the following IVP:

$$y' - y = 2te^{-t}, \qquad y(0) = 1$$

Solution. 1)

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{2te^{-t}\}$$

Using linearity of the Laplace transform,

$$\begin{aligned}\mathcal{L}\{y'\} - \mathcal{L}\{y\} &= 2\mathcal{L}\{te^{-t}\}\\ sY(s) - y(0) - Y(s) &= 2\frac{1}{(s+1)^2}\end{aligned}$$

2) Substitute the initial condition $y(0) = 1$:

$$\begin{aligned} sY(s) - 1 - Y(s) &= \frac{2}{(s+1)^2} \\ Y(s)(s-1) &= 1 + \frac{2}{(s+1)^2} \end{aligned}$$

3)

$$Y(s) = \frac{1}{s-1} + \frac{2}{(s-1)(s+1)^2}$$

4) Now take the inverse Laplace transform.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-1)(s+1)^2}\right\} \end{aligned}$$

Use partial fractions:

$$\frac{1}{(s-1)(s+1)^2} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Solving gives

$$A = \frac{1}{2}, \qquad B = -\frac{1}{2}, \qquad C = -1$$

Thus

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{2(s-1)} - \frac{1}{2(s+1)} - \frac{1}{(s+1)^2}\right\}$$

Using the inverse Laplace table (see Appendix B):

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{(s+a)^2}\right\} = te^{-at}$$

we obtain

$$y(t) = e^t + \frac{1}{2}e^t - \frac{1}{2}e^{-t} - te^{-t}$$

$$\boxed{y(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t} - te^{-t}}$$

as desired. □

Solving IVPs with Discontinuous Input

Example 4.11. Solve the following IVP:

$$y'' + 3y' + 2y = g(t), \qquad y(0) = 1, y'(0) = -1$$

where $g(t) = H\left(t - \frac{\pi}{2}\right)\sin(t)$.

Solution. 1)

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{H(t - \tfrac{\pi}{2})\sin t\}$$

2), 3) Using Laplace transforms of derivatives,

$$s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos t\}$$

Using the initial conditions $y(0) = 1$, $y'(0) = -1$,

$$(s^2 + 3s + 2)Y(s) - s + 1 - 3 = e^{-\frac{\pi}{2}s}\frac{s}{s^2+1}$$

Thus

$$Y(s) = \frac{s+2}{s^2+3s+2} + e^{-\frac{\pi}{2}s}\frac{s}{(s^2+1)(s^2+3s+2)}$$

Factor the quadratic

$$s^2 + 3s + 2 = (s+1)(s+2)$$

so

$$Y(s) = \frac{s+2}{(s+1)(s+2)} + e^{-\frac{\pi}{2}s}\frac{s}{(s^2+1)(s+1)(s+2)}$$

4) Take the inverse Laplace transform.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)(s+2)}\right\} + \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{s}{(s^2+1)(s+1)(s+2)}\right\}$$

Since

$$\frac{s+2}{(s+1)(s+2)} = \frac{1}{s+1},$$

we obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

Next perform partial fractions:

$$\frac{s}{(s^2+1)(s+1)(s+2)} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{s+2}$$

Solving gives

$$A = \frac{1}{10}, \qquad B = \frac{3}{10}, \qquad C = -\frac{1}{2}, \qquad D = \frac{2}{5}.$$

Thus

$$\begin{aligned} y(t) = e^{-t} &+ \frac{1}{10}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{s}{s^2+1}\right\} + \frac{3}{10}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{1}{s^2+1}\right\} \\ &- \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{1}{s+1}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{1}{s+2}\right\}. \end{aligned}$$

Using the shift property

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = H(t-c)f(t-c),$$

we obtain

$$\begin{aligned} y(t) = e^{-t} &+ \frac{1}{10} H\Big(t - \frac{\pi}{2}\Big)\cos\Big(t - \frac{\pi}{2}\Big) + \frac{3}{10} H\Big(t - \frac{\pi}{2}\Big)\sin\Big(t - \frac{\pi}{2}\Big) \\ &- \frac{1}{2} H\Big(t - \frac{\pi}{2}\Big) e^{-(t-\frac{\pi}{2})} + \frac{2}{5} H\Big(t - \frac{\pi}{2}\Big) e^{-2(t-\frac{\pi}{2})}. \end{aligned}$$

Factoring $H(t - \frac{\pi}{2})$ gives

$$\boxed{y(t) = e^{-t} + H\Big(t - \frac{\pi}{2}\Big)\left[\frac{1}{10}\sin t - \frac{3}{10}\cos t - \frac{1}{2}e^{-(t-\frac{\pi}{2})} + \frac{2}{5}e^{-2(t-\frac{\pi}{2})}\right]}$$

□

Laplace Transform of a Periodic Function with Jump Discontinuities

Theorem 4.5.

Suppose

$$f(t + nT) = f(t), \qquad n = 1, 2, 3, \ldots$$

For a periodic function with period T, the Laplace transform is

$$\mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{f_1(t)\}}{1 - e^{-sT}}$$

where $f_1(t)$ is the function over one cycle (one period) of the periodic function.

Example 4.12. Find the LP for function $f(t) = \begin{cases} 1 & 2na \le t < (2n+1)a \\ 0 & (2n+1)a \le t < 2(n+1)a \end{cases}$ for $n \in \mathbb{Z}$. Note that f is a periodic function with period of $2a$.

Solution. We start by noting that

$$\begin{aligned} f_1(t) &= (1)[H(t) - H(t-a)] + (-1)[H(t-a) - H(t-2a)] \\ &= H(t) - 2H(t-a) + H(t-2a) \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{L}\{f_1(t)\} &= \mathcal{L}\{H(t)\} - 2\mathcal{L}\{H(t-a)\} + \mathcal{L}\{H(t-2a)\} \\ &= \frac{1}{s} - \frac{2e^{-as}}{s} + \frac{e^{-2as}}{s} \\ &= \frac{1}{s}(1 - 2e^{-as} + e^{-2as}) \\ &= \frac{1}{s}(1 - e^{-as})^2 \end{aligned}$$

For a periodic function with period $T = 2a$,

$$\mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{f_1(t)\}}{1 - e^{-sT}} = \frac{\frac{1}{s}(1 - e^{-as})^2}{1 - e^{-2as}}$$

We also know that

$$1 - e^{-2as} = (1 - e^{-as})(1 + e^{-as})$$

so

$$\boxed{\mathcal{L}\{f(t)\} = \frac{1}{s}\frac{1 - e^{-as}}{1 + e^{-as}}}$$

as desired. □

Solving IVPs with Impulse Input

Definition 4.2. **[Dirac Delta Function]**

The Dirac delta function $\delta(t)$ is defined by

$$\delta(t) = 0 \quad \text{for } t \neq 0$$

and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Discovery 4.1. Dirac Delta function satisfies the **sifting property**

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - a)\, dt = f(a).$$

Note 4.2. We have

$$\begin{aligned} &\mathcal{L}\{\delta(t)\} = 1, \qquad \mathcal{L}\{\delta(t-a)\} = e^{-as}, a \geq 0 \\ &\mathcal{L}\{\delta(t-a)f(t)\} = e^{-as}f(a) \end{aligned}$$

Example 4.13. Consider the motion of a simple pendulum with no damping and natural frequency $\omega_0 = 1$. The pendulum is initially at rest with $y(0) = 1$, and is given a kick at $t = \frac{\pi}{2}$ by an impulse of magnitude A per unit mass.
Find the response of the system to this input.

Solution.

$$y'' + \omega_0^2 y(t) = A\, \delta\left(t - \frac{\pi}{2}\right), \qquad y(0) = 1, \quad y'(0) = 0, \quad {\omega_0}^2 = 1$$

Thus we have

$$\begin{aligned}\mathcal{L}\{y''\} + \mathcal{L}\{y(t)\} &= \mathcal{L}\left\{A\delta\Big(t - \frac{\pi}{2}\Big)\right\} \\ s^2Y(s) - sy(0) - y'(0) + Y(s) &= Ae^{-\frac{\pi}{2}s}\end{aligned}$$

Substitute the initial conditions:

$$\begin{aligned}s^2Y(s) - s + Y(s) &= Ae^{-\frac{\pi}{2}s} \\ (s^2+1)Y(s) - s &= Ae^{-\frac{\pi}{2}s} \\ Y(s) &= \frac{s}{s^2+1} + Ae^{-\frac{\pi}{2}s}\frac{1}{s^2+1}\end{aligned}$$

Taking the inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + A\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{1}{s^2+1}\right\}$$

Using

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t,$$

and the shift property,

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)f(t-a),$$

we obtain

$$y(t) = \cos t + A\,H\Big(t - \frac{\pi}{2}\Big)\sin\Big(t - \frac{\pi}{2}\Big).$$

Since

$$\sin\Big(t - \frac{\pi}{2}\Big) = -\cos t,$$

the response simplifies to

$$\boxed{y(t) = \cos t - A\,H\Big(t - \frac{\pi}{2}\Big)\cos t}.$$

as desired. □

4.5 Convolution

Definition 4.3. **[Convolution]**

For any functions $f(t)$ and $g(t)$ that are piecewise continuous on $0 \le t \le T$, the **convolution** is

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Proposition 4.5. For $f(t)$ and $g(t)$, we have

$$(f * g)(t) = (g * f)(t)$$

Theorem 4.6. Convolution Theorem

If $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$ both exist for all $s > a > 0$, then

$$F(s)G(s) = \mathcal{L}\{(f * g)(t)\},$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

Example 4.14. Use the convolution theorem to find the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}.$$

Solution. We first factor the expression as a product:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right\}.$$

Let

$$F(s) = \frac{1}{s^2}, \qquad G(s) = \frac{1}{s^2+1}.$$

Then by the convolution theorem, $\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$, where

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t,$$

and

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = (f * g)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t (t-\tau)\sin\tau\,d\tau.$$

We evaluate this integral by integration by parts. Take

$$u = t - \tau, \qquad dv = \sin\tau\,d\tau.$$

Then

$$du = -\,d\tau, \qquad v = -\cos\tau.$$

Hence,

$$\begin{aligned} I &= uv\Big|_0^t - \int_0^t v\,du \\ &= -(t-\tau)\cos\tau\Big|_0^t - \int_0^t \cos\tau\,d\tau. \end{aligned}$$

Now evaluate each term separately. For the boundary term,

$$-(t-\tau)\cos\tau\Big|_0^t = -\big[(t-t)\cos t - (t-0)\cos 0\big] = t.$$

Also,

$$\int_0^t \cos\tau\, d\tau = \sin\tau\Big|_0^t = \sin t.$$

So

$$I = t - \sin t.$$

Therefore,

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t - \sin t}.$$

as desired. □

Exercise 4.5. Use partial fraction decomposition to find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}.$$

4.6 Laplace Transform of Integral

We begin with the observation that convolving a function with the constant function 1 gives its integral:

$$(f*1)(t) = (1*f)(t) = \int_0^t (1)\, f(\tau)\, d\tau = \int_0^t f(\tau)\, d\tau.$$

Define

$$I_f(t) = \int_0^t f(\tau)\, d\tau.$$

Then, by the Fundamental Theorem of Calculus,

$$I_f'(t) = f(t).$$

Now take Laplace transforms on both sides:

$$\mathcal{L}\{I_f'(t)\} = \mathcal{L}\{f(t)\}.$$

Using the differentiation formula for Laplace transforms,

$$\mathcal{L}\{I_f'(t)\} = s\,\mathcal{L}\{I_f(t)\} - I_f(0).$$

Since

$$I_f(0) = \int_0^0 f(\tau)\, d\tau = 0,$$

we get

$$s\,\mathcal{L}\{I_f(t)\} = \mathcal{L}\{f(t)\}.$$

Therefore,

$$\mathcal{L}\{I_f(t)\} = \frac{1}{s}\,\mathcal{L}\{f(t)\}.$$

If we write

$$F(s) = \mathcal{L}\{f(t)\},$$

then this becomes

$$\mathcal{L}\{I_f(t)\} = \frac{1}{s}F(s).$$

Taking inverse Laplace transforms gives

$$I_f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

Now let

$$G(s) = \frac{1}{s}.$$

Then

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \qquad \mathcal{L}^{-1}\{G(s)\} = 1.$$

By the convolution theorem,

$$\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \mathcal{L}^{-1}\{G(s)F(s)\} = (g * f)(t).$$

Since $g(t) = 1$, we obtain

$$(g * f)(t) = (1 * f)(t) = \int_0^t f(\tau)\,d\tau.$$

Hence,

$$\boxed{\int_0^t f(\tau)\,d\tau = (1 * f)(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}}.$$

Example 4.15. Solve the integral equation

$$y(t) = t + \int_0^t \sin(t - z)\,y(z)\,dz.$$

Solution. The integral term is a convolution, since

$$(f * g)(t) = \int_0^t f(t - z)\,g(z)\,dz.$$

Taking

$$f(t) = \sin t, \qquad g(t) = y(t),$$

we get

$$\int_0^t \sin(t - z)\,y(z)\,dz = (\sin * y)(t).$$

So the equation becomes

$$y(t) = t + (\sin * y)(t).$$

Let

$$Y(s) = \mathcal{L}\{y(t)\}.$$

Taking Laplace transforms gives

$$Y(s) = \mathcal{L}\{t\} + \mathcal{L}\{(\sin * y)(t)\}.$$

Using

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \qquad \mathcal{L}\{\sin t\} = \frac{1}{s^2+1},$$

and the convolution theorem,

$$\mathcal{L}\{(\sin * y)(t)\} = \mathcal{L}\{\sin t\}\mathcal{L}\{y(t)\} = \frac{1}{s^2+1}Y(s),$$

we obtain

$$Y(s) = \frac{1}{s^2} + \frac{1}{s^2+1}Y(s).$$

Now solve for $Y(s)$:

$$Y(s) - \frac{1}{s^2+1}Y(s) = \frac{1}{s^2},$$

so

$$\left(1 - \frac{1}{s^2+1}\right)Y(s) = \frac{1}{s^2}.$$

Since

$$1 - \frac{1}{s^2+1} = \frac{s^2}{s^2+1},$$

it follows that

$$\frac{s^2}{s^2+1}Y(s) = \frac{1}{s^2},$$

and hence

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}.$$

Taking inverse Laplace transforms term by term,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = t + \frac{t^3}{6}.$$

Therefore,

$$\boxed{y(t) = t + \frac{t^3}{6}}.$$

□

5 System of Ordinary Differential Equations

Definition 5.1. **[System of]**

An n-dimensional system of ODEs is a set of n equations for n unknown functions (n dependent variables) of one independent variable.

Definition 5.2. **[Order (SDE)]**

The **order** of the SDE is the order of the highest derivative appearing in the system.

Definition 5.3. **[Linearity (SDE)]**

The SDE is **linear** if all equations are linear; otherwise, the SDE is *nonlinear*.

Theorem 5.1.

Any higher-order ODE can be expressed as an equivalent system of first-order ODEs.

Consider the n-th order ODE

$$y^{(n)}(x) = f\big(x, y, y', \dots, y^{(n-1)}\big).$$

Define new variables by

$$y_1 = y, \qquad y_2 = y', \qquad y_3 = y'', \qquad \dots, \qquad y_n = y^{(n-1)}.$$

Then

$$\begin{aligned} y_1' &= y' = y_2, \\ y_2' &= y'' = y_3, \\ y_3' &= y''' = y_4, \\ &\vdots \\ y_{n-1}' &= y^{(n-1)} = y_n, \end{aligned}$$

and

$$y_n' = y^{(n)} = f\big(x, y, y', \dots, y^{(n-1)}\big) = f(x, y_1, y_2, \dots, y_n).$$

Hence the original n-th order ODE is equivalent to the first-order system

$$\begin{cases} y_1 = y, \\ y_2 = y', \\ y_3 = y'' \\ \vdots \\ y_n = y^{(n-1)} \end{cases} \Rightarrow \begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ \vdots \\ y_{n-1}' = y_n, \\ y_n' = f(x, y_1, y_2, \dots, y_n). \end{cases}$$

General form of a system of differential equations (SDE) with n equations:

$$\begin{cases} y_1' = a_{11}(x)y_1 + \cdots + a_{1n}(x)y_n + F_1(x), \\ y_2' = a_{21}(x)y_1 + \cdots + a_{2n}(x)y_n + F_2(x), \\ \vdots \\ y_n' = a_{n1}(x)y_1 + \cdots + a_{nn}(x)y_n + F_n(x). \end{cases}$$

In vector form,

$$\vec{y}\,'(x) = A(x)\vec{y}(x) + \vec{F}(x).$$

Example 5.1. Consider the SDE:

$$\begin{cases} \dfrac{dx}{dt} = ax + by + f(t), \qquad x(t_0) = x_0, \\ \dfrac{dy}{dt} = cx + dy + g(t), \qquad y(t_0) = y_0. \end{cases}$$

1. Write the IVP as a vector DE.
2. Convert the SDE into a 2nd-order DE.

Solution.

1. Let

$$\vec{X}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \vec{X}\,'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}, \qquad \vec{F}(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

   Hence, $\vec{X}\,'(t) = A\vec{X}(t) + \vec{F}(t)$.

2. From equation (1),

$$x' = ax + by + f(t) \quad \Longrightarrow \quad y = \frac{1}{b}x' - \frac{a}{b}x - \frac{1}{b}f(t).$$

   Differentiating to obtain y', then substitute into equation (2), we get,

$$\frac{1}{b}x'' - \frac{a}{b}x' - \frac{1}{b}f'(t) = cx + d\left(\frac{1}{b}x' - \frac{a}{b}x - \frac{1}{b}f(t)\right) + g(t).$$

   Multiplying by b, we get

$$x'' - (a+d)x' + (ad - bc)x = f'(t) - df(t) + bg(t).$$

   This is a linear 2nd-order DE with constant coefficients. So we first solve for $x(t)$, then return to the system to find $y(t)$:

$$y = \frac{1}{b}x' - \frac{a}{b}x - \frac{1}{b}f(t).$$

   Thus we obtain the two solutions $x(t)$ and $y(t)$ for the system.

Example 5.2. Use the method of elimination to solve the SDE:

$$\begin{cases} x' + 2x - y' = 4t, & (1) \\ y' - 6x + y = 0, & (2) \end{cases}$$

From equation (2), we have $y' = 6x - y$.

Solution. Add equations (1) and (2):

$$x' - 4x + y = 4t,$$

so

$$y = 4t - x' + 4x.$$

From

$$x' = 4x - y + 4t,$$

differentiate to get

$$x'' = 4x' - y' + 4.$$

Using

$$y' = 6x - y$$

and

$$y = 4t - x' + 4x,$$

we obtain

$$x'' = 4x' - \big(6x - (4t - x' + 4x)\big) + 4.$$

Hence,

$$x'' - 3x' + 2x = 4t + 4.$$

This is a linear 2nd-order DE with constant coefficients. Solving for $x(t)$, we get

$$x(t) = c_1 e^{2t} + c_2 e^t + 2t + 5.$$

Now find $y(t)$ from

$$y = 4t - x' + 4x.$$

Substituting $x(t)$ and $x'(t)$, we get

$$y(t) = 2c_1 e^{2t} + 3c_2 e^t + 12t + 18.$$

□

5.1 Theory of linear SDEs

Any n-th order ODE can be rewritten as an n-dimensional system of first-order ODEs.

Theorem 5.2. Existence and Uniqueness (E/U) Theorem.

Consider the vector IVP

$$\vec{y}\,'(x) = A(x)\vec{y}(x) + \vec{F}(x), \qquad \vec{y}(x_0) = \vec{y}_0.$$

Suppose $A(x)$ and $\vec{F}(x)$ are continuous on an open interval I, and $x_0 \in I$. Then for any vector $\vec{y}_0 \in \mathbb{R}^n$, the vector IVP has a unique solution on I.

Theorem 5.3.

If $\vec{u}(x)$ is a solution to the homogeneous vector DE

$$\vec{y}\,'(x) = A\vec{y}(x) \quad \text{on } I_1,$$

and $\vec{v}(x)$ is a solution to the inhomogeneous vector DE

$$\vec{y}\,'(x) = A\vec{y}(x) + \vec{F}(x) \quad \text{on } I_2,$$

then

$$\vec{y}(x) = \vec{u}(x) + \vec{v}(x)$$

is a solution of

$$\vec{y}\,'(x) = A\vec{y}(x) + \vec{F}(x) \quad \text{on } I = I_1 \cap I_2.$$

Proof. Let

$$\vec{y}(x) = \vec{u}(x) + \vec{v}(x).$$

Then

$$\vec{y}\,'(x) = \vec{u}\,'(x) + \vec{v}\,'(x).$$

Since

$$\vec{u}\,'(x) = A\vec{u}(x), \qquad \vec{v}\,'(x) = A\vec{v}(x) + \vec{F}(x),$$

we get

$$\vec{y}\,'(x) = A\vec{u}(x) + A\vec{v}(x) + \vec{F}(x) = A\big(\vec{u}(x) + \vec{v}(x)\big) + \vec{F}(x).$$

Hence

$$\vec{y}\,'(x) = A\vec{y}(x) + \vec{F}(x).$$

So $\vec{y}(x)$ satisfies the inhomogeneous vector DE, and it is called the general solution of the vector DE. $\square$

5.1.1 Principle of Superpositions

Theorem 5.4. Principle of Superposition

If $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n$ are solutions of the homogeneous vector DE

$$\vec{y}\,'(x) = A\vec{y}(x)$$

on an interval I, then

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + \cdots + c_n\vec{y}_n$$

is also a solution on I for any constants

$$c_1, c_2, \ldots, c_n.$$

Definition 5.4. **[Linear Independence]**

The set of vector functions

$$\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n$$

is **linearly independent** on I if

$$\alpha_1\vec{y}_1 + \alpha_2\vec{y}_2 + \cdots + \alpha_n\vec{y}_n = \vec{0} \qquad \text{for all } x \in I$$

implies

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Otherwise, they are linearly dependent.

5.1.2 Wronskian and Linearly Independent Solutions

Lecture 20 - Tuesday, March 17

Proposition 5.1. Wronskian Proposition

Let $A(x)$ be a continuous function $I \to \mathbb{R}^{n\times n}$. Let $\vec{Y}_1, \vec{Y}_2, \ldots, \vec{Y}_n$ be solutions of the homogeneous equation

$$\vec{Y}\,'(x) = A(x)\vec{Y}(x) \qquad \text{on } I.$$

Then $\vec{Y}_1, \vec{Y}_2, \ldots, \vec{Y}_n$ are linearly independent on I if and only if

$$W[\vec{Y}_1, \vec{Y}_2, \ldots, \vec{Y}_n] \neq 0,$$

that is,

$$\begin{vmatrix} Y_{11}(x) & Y_{12}(x) & \cdots & Y_{1n}(x) \\ Y_{21}(x) & Y_{22}(x) & \cdots & Y_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n1}(x) & Y_{n2}(x) & \cdots & Y_{nn}(x) \end{vmatrix} \neq 0 \qquad \text{for all } x \in I.$$

Theorem 5.5. Abel's Formula

Suppose the conditions of the E/U theorem are satisfied and $\vec{Y}_1, \vec{Y}_2, \ldots, \vec{Y}_n$ are the solutions to the homogeneous DE

$$\vec{Y}'(x) = A\vec{Y}(x) \qquad \text{on } I.$$

Then for any $x_0 \in I$, the Wronskian of these functions satisfies

$$W[\vec{Y}_1, \vec{Y}_2, \ldots, \vec{Y}_n](x) = W(x_0) \exp\left(-\int_{x_0}^{x} \operatorname{tr}(A(s))\, ds\right).$$

Suppose that $A(x)$ is an $n \times n$ matrix function and continuous on I. To solve the system of linear 1st order ODEs

$$\vec{Y}'(x) = A(x)\vec{Y}(x) + \vec{F}(x) \qquad \text{on } I,$$

proceed as follows:

1. Find n linearly independent homogeneous solutions

   $$\vec{Y}_1, \vec{Y}_2, \ldots, \vec{Y}_n$$

   to the homogeneous equation

   $$\vec{Y}'(x) = A(x)\vec{Y}(x).$$

   Then

   $$\vec{Y}_h(x) = c_1\vec{Y}_1(x) + c_2\vec{Y}_2(x) + \cdots + c_n\vec{Y}_n(x),$$

   where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

2. Find a particular solution $\vec{Y}_p(x)$ to the inhomogeneous vector DE

   $$\vec{Y}'(x) = A(x)\vec{Y}(x) + \vec{F}(x).$$

3. The general solution is

   $$\vec{Y}(x) = \vec{Y}_h(x) + \vec{Y}_p(x),$$

   i.e.

   $$\vec{Y}(x) = c_1\vec{Y}_1(x) + c_2\vec{Y}_2(x) + \cdots + c_n\vec{Y}_n(x) + \vec{Y}_p(x).$$

5.2 Solving homogeneous vector DEs with constant coefficients

Consider

$$\vec{Y}'(x) = A\vec{Y}(x),$$

where A is an $n \times n$ constant matrix.

Discovery 5.1. Recall from ODEs: if

$$x'(t) = ax(t),$$

then the solution is

$$x(t) = ke^{at}.$$

If we try

$$\vec{Y}(x) = \vec{v}\, e^{\lambda x},$$

then

$$\vec{Y}'(x) = \lambda \vec{v}\, e^{\lambda x}.$$

Here the n-dimensional vector $\vec{v}$ and constant λ need to be determined. Substituting into the vector DE:

$$\text{LHS: } \vec{Y}'(x) = \lambda \vec{v}\, e^{\lambda x}, \qquad \text{RHS: } A(\vec{v}\, e^{\lambda x}) = (A\vec{v})e^{\lambda x}.$$

So

$$\lambda \vec{v}\, e^{\lambda x} = (A\vec{v})e^{\lambda x}.$$

Since $e^{\lambda x} \neq 0$, we get

$$\lambda \vec{v} = A\vec{v},$$

or equivalently

$$(A - \lambda I)\vec{v} = \vec{0}.$$

To have a nontrivial solution, we must have

$$\boxed{\det(A - \lambda I) = 0,}$$

which is the characteristic equation.

Note 5.1. We will have a nontrivial solution if λ is the eigenvalue and $\vec{v}$ is the corresponding eigenvector for the solution.

Theorem 5.6.

Suppose A is a 2×2 constant matrix and has eigenvalues λ_1 and λ_2. We have 3 cases:

1. $\lambda_1 \neq \lambda_2$, real distinct eigenvalues.

   Then there are two linearly independent eigenvectors $\vec{v}_1$ and $\vec{v}_2$, and

   $$\vec{Y}_h(x) = c_1 \vec{v}_1 e^{\lambda_1 x} + c_2 \vec{v}_2 e^{\lambda_2 x}.$$

2. $\lambda_1 = \lambda_2 = \lambda$, repeated eigenvalue.

   Then one solution is

   $$\vec{Y}_1(x) = \vec{v} e^{\lambda x}.$$

   For a second solution,

   $$\vec{Y}_2(x) = e^{\lambda x}(x\vec{v} + \vec{w}),$$

   where $\vec{w}$ satisfies

   $$(A - \lambda I)\vec{w} = \vec{v}.$$

   Hence

   $$\vec{Y}_h(x) = c_1 \vec{v} e^{\lambda x} + c_2 e^{\lambda x}(x\vec{v} + \vec{w}).$$

3. $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, complex eigenvalues.

   If

   $$\vec{v}_1 = \vec{u} + i\vec{w}, \qquad \vec{v}_2 = \vec{u} - i\vec{w},$$

   then we will see the solution to this case in an example.

Example 5.3. Solve the system of equations:

$$\begin{cases} x'(t) = 2x + 3y, \\ y'(t) = 2x + y. \end{cases}$$

Let

$$\vec{X}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \vec{X}\,'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}.$$

so we have

$$\vec{X}'(t) = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \vec{X}$$

Solution.

$$\vec{X}\,'(t) = A\vec{X}(t), \qquad A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}.$$

The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = 0.$$

Hence

$$(\lambda - 4)(\lambda + 1) = 0,$$

so the eigenvalues are

$$\lambda_1 = 4, \qquad \lambda_2 = -1.$$

For $\lambda_1 = 4$,

$$A - 4I = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix},$$

so an eigenvector is

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

For $\lambda_2 = -1$,

$$A + I = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix},$$

so an eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore the general solution is

$$\vec{X}(t) = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}.$$

Equivalently,

$$x(t) = 3c_1 e^{4t} + c_2 e^{-t}, \qquad y(t) = 2c_1 e^{4t} - c_2 e^{-t}.$$

□

Example 5.4. Solve

$$\begin{cases} x'(t) = -2x + y, \\ y'(t) = -3x - 4y. \end{cases}$$

Solution. Let

$$\vec{X}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \vec{X}\,'(t) = A\vec{X}(t), \qquad A = \begin{bmatrix} -2 & 1 \\ -3 & -4 \end{bmatrix}.$$

The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = (-2-\lambda)(-4-\lambda) + 3 = \lambda^2 + 6\lambda + 11 = 0.$$

Hence

$$\lambda = \frac{-6 \pm \sqrt{36-44}}{2} = -3 \pm i.$$

For $\lambda_1 = -3+i$, solve $(A - \lambda_1 I)\vec{v} = \vec{0}$:

$$A - (-3+i)I = \begin{bmatrix} 1-i & 1 \\ -3 & -1-i \end{bmatrix}.$$

An eigenvector is

$$\vec{v} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus

$$\vec{X}_c(t) = e^{(-3+i)t}\vec{v} = e^{-3t}(\cos t + i \sin t)(\vec{u} + i\vec{w}),$$

where

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Taking real and imaginary parts gives two real solutions:

$$\vec{X}_1(t) = e^{-3t}\left(\vec{u}\cos t - \vec{w}\sin t\right), \qquad \vec{X}_2(t) = e^{-3t}\left(\vec{u}\sin t + \vec{w}\cos t\right).$$

Therefore the general solution is

$$\vec{X}(t) = c_1 e^{-3t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right) + c_2 e^{-3t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t\right).$$

Equivalently,

$$x(t) = e^{-3t}\big(c_1\cos t + c_2 \sin t\big),$$
$$y(t) = e^{-3t}\big((-c_1 + c_2)\cos t - (c_1 + c_2)\sin t\big).$$

□

Note 5.2. If we use $\lambda_2 = -3 - i\sqrt{2}$ and find $\vec{v}_2$, the real parts of the solution will be the same as $\vec{A}_1(t), \vec{A}_2(t)$.

Exercise 5.1. Show that you find the same $\vec{A}_1(t)$ and $\vec{A}_2(t)$ using $\lambda_2 = -3 - i\sqrt{2}$.

Example 5.5. Solve

$$\vec{Y}\,'(x) = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}\vec{Y}(x).$$

Solution. Let

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}.$$

First find the eigenvalues:

$$\det(A - \lambda I) = 0 \quad \Longrightarrow \quad \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0.$$

So

$$(4-\lambda)(2-\lambda) + 1 = 0$$

which gives

$$\lambda^2 - 6\lambda + 9 = 0 \quad \Longrightarrow \quad (\lambda - 3)^2 = 0.$$

Hence

$$\lambda_1 = \lambda_2 = 3.$$

Now find an eigenvector for $\lambda = 3$:

$$(A - 3I)\vec{v} = \vec{0} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus

$$v_1 + v_2 = 0 \quad \Longrightarrow \quad v_2 = -v_1.$$

Choose $v_1 = 1$, so

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore one solution is

$$\vec{Y}_1(x) = e^{3x} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For a second solution, use

$$\vec{Y}_2(x) = e^{3x}(x\vec{v} + \vec{w}),$$

where $\vec{w}$ satisfies

$$(A - 3I)\vec{w} = \vec{v}.$$

So

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This gives

$$w_1 + w_2 = 1.$$

Choose $w_1 = 0,\ w_2 = 1$, so

$$\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence

$$\vec{Y}_2(x) = e^{3x} \left(x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

Therefore the general solution is

$$\vec{Y}(x) = c_1 e^{3x} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3x} \left(x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

□

Question 5.1.

How to find $\vec{Y}_p(x)$?

Special case: If $\vec{F}(x) = \vec{b}$, a constant vector, then

$$\vec{Y}\,'(x) = A\vec{Y}(x) + \vec{F}(x), \qquad \text{where } \vec{F}(x) = \vec{b}.$$

The general solution is

$$\vec{Y}(x) = \vec{Y}_h(x) - A^{-1}\vec{b}.$$

So a particular solution is

$$\vec{Y}_p = -A^{-1}\vec{b}.$$

Example 5.6. Solve the IVP:

$$\begin{cases} \dfrac{dD}{dt} = 80 - 4p(t), \quad D(0) = 70, \\ \dfrac{dp}{dt} = D(t) - 100, \quad p(0) = 20. \end{cases}$$

Solution. Rewrite the system as

$$\begin{cases} \dfrac{dD}{dt} = 0 \cdot D(t) - 4p(t) + 80, \\ \dfrac{dp}{dt} = D(t) + 0 \cdot p(t) - 100. \end{cases}$$

Let

$$\vec{Y}(t) = \begin{bmatrix} D(t) \\ p(t) \end{bmatrix}, \qquad \vec{F}(t) = \begin{bmatrix} 80 \\ -100 \end{bmatrix}, \qquad \vec{Y}(0) = \begin{bmatrix} 70 \\ 20 \end{bmatrix}.$$

Also let

$$A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}, \qquad \vec{Y}\,'(t) = \begin{bmatrix} D'(t) \\ p'(t) \end{bmatrix}.$$

Then the system becomes

$$\vec{Y}\,'(t) = A\vec{Y}(t) + \vec{F}(t).$$

□

Exercise 5.2. Show that

$$\vec{Y}_h(t) = c_1 \begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} 2\sin(2t) \\ -\cos(2t) \end{bmatrix}.$$

To find $\vec{Y}_p(t)$, we use the formula

$$\vec{Y}_p(t) = -A^{-1}\vec{b}.$$

5.2.1 Undetermined Coefficients

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If all components of $\vec{F}(x)$ are suitable for the method of undetermined coefficients, $\Rightarrow$ we can use this method.

Example 5.7. If

$$\vec{F}(x) = \begin{bmatrix} \sin x - 3x \\ e^{2x} + 1 \end{bmatrix},$$

then the form for undetermined coefficients is

$$A\sin x + B\cos x + Cx + D + Ee^{2x}.$$

Hence we try

$$\vec{Y}_p(x) = \begin{bmatrix} A_1 \sin x + B_1 \cos x + C_1 x + D_1 + E_1 e^{2x} \\ A_2 \sin x + B_2 \cos x + C_2 x + D_2 + E_2 e^{2x} \end{bmatrix}.$$

Then substitute into the vector DE to determine these constants.

Exercise 5.3. Solve the IVP:

$$\begin{cases} \dfrac{dD}{dt} = 80 - 4p(t), \quad D(0) = 70, \\ \dfrac{dp}{dt} = D(t) - 100, \quad p(0) = 20. \end{cases}$$

using the method of undertermined coefficients.

Example 5.8. Solve

$$\vec{x}\,'(t) = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}.$$

Solution. Trial solution:

$$\vec{x}_p(t) = \begin{bmatrix} A_1 e^{-2t} + B_1 e^{t} \\ A_2 e^{-2t} + B_2 e^{t} \end{bmatrix}.$$

Substituting into the DE:

$$\vec{x}_p\,'(t) = \begin{bmatrix} -2A_1 e^{-2t} + B_1 e^{t} \\ -2A_2 e^{-2t} + B_2 e^{t} \end{bmatrix}.$$

Since

$$\vec{x}\,'(t) = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix},$$

we get

$$\begin{bmatrix} -2A_1e^{-2t} + B_1e^{t} \\ -2A_2e^{-2t} + B_2e^{t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} A_1e^{-2t} + B_1e^{t} \\ A_2e^{-2t} + B_2e^{t} \end{bmatrix} + \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}.$$

That is,

$$\begin{bmatrix} -2A_1e^{-2t} + B_1e^{t} \\ -2A_2e^{-2t} + B_2e^{t} \end{bmatrix} = \begin{bmatrix} (A_1 + A_2 + 1)e^{-2t} + (B_1 + B_2)e^{t} \\ (4A_1 - 2A_2)e^{-2t} + (4B_1 - 2B_2 - 2)e^{t} \end{bmatrix}.$$

So

$$\begin{cases} -2A_1e^{-2t} + B_1e^{t} = (A_1 + A_2 + 1)e^{-2t} + (B_1 + B_2)e^{t}, \\ -2A_2e^{-2t} + B_2e^{t} = (4A_1 - 2A_2)e^{-2t} + (4B_1 - 2B_2 - 2)e^{t}. \end{cases}$$

Hence

$$B_2 = 0, \qquad A_1 = 0, \qquad A_2 = -1, \qquad B_1 = \frac{1}{2}.$$

Therefore

$$\vec{x}_p(t) = \begin{bmatrix} A_1e^{-2t} + B_1e^{t} \\ A_2e^{-2t} + B_2e^{t} \end{bmatrix} = \begin{bmatrix} 0 + \frac{1}{2}e^{t} \\ -1 \cdot e^{-2t} + 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{t} \\ -e^{-2t} \end{bmatrix}.$$

The general solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t),$$

that is,

$$\vec{x}(t) = c_1e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}e^{t} \\ -e^{-2t} \end{bmatrix}.$$

as desired. □

5.2.2 Variation of Parameters

Suppose the problem is of the form:

$$\vec{Y}\,'(t) = A\vec{Y}(t) + \vec{F}(t)$$

1. Find the homogeneous solution to the homogeneous vector system

   $$\vec{x}\,'(t) = A\vec{x}(t).$$

   Then

   $$\vec{Y}_h(t) = c_1\vec{Y}_1(t) + c_2\vec{Y}_2(t).$$

2. For a particular solution, let

   $$\vec{Y}_p(t) = u_1(t)\vec{Y}_1(t) + u_2(t)\vec{Y}_2(t).$$

3. Substitute into the DE:

$$\vec{Y}_p\,'(t) = u_1'(t)\vec{Y}_1(t) + u_1(t)\vec{Y}_1'(t) + u_2'(t)\vec{Y}_2(t) + u_2(t)\vec{Y}_2'(t).$$

Since $\vec{Y}_p$ must satisfy

$$\vec{Y}_p'(t) = A\vec{Y}_p(t) + \vec{F}(t),$$

we get

$$u_1'(t)\vec{Y}_1(t) + u_1(t)\vec{Y}_1'(t) + u_2'(t)\vec{Y}_2(t) + u_2(t)\vec{Y}_2'(t) = A\big(u_1(t)\vec{Y}_1(t) + u_2(t)\vec{Y}_2(t)\big) + \vec{F}(t).$$

Using

$$\vec{Y}_1'(t) = A\vec{Y}_1(t), \qquad \vec{Y}_2'(t) = A\vec{Y}_2(t),$$

this becomes

$$u_1'(t)\vec{Y}_1(t) + A\,u_1(t)\vec{Y}_1(t) + u_2'(t)\vec{Y}_2(t) + A\,u_2(t)\vec{Y}_2(t) = A\,u_1(t)\vec{Y}_1(t) + A\,u_2(t)\vec{Y}_2(t) + \vec{F}(t).$$

Cancelling the matching terms gives

$$\boxed{u_1'(t)\vec{Y}_1(t) + u_2'(t)\vec{Y}_2(t) = \vec{F}(t)}.$$

We are looking for $u_1(t)$ and $u_2(t)$ that satisfy this condition.

This way, we obtain $\vec{Y}_p(t)$, so the general solution is

$$\vec{Y}(t) = \vec{Y}_h(t) + \vec{Y}_p(t).$$

Example 5.9. Solve

$$\vec{Y}\,'(t) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\vec{Y}(t) + \begin{bmatrix} \frac{1}{t} \\ 5 + \frac{2}{t} \end{bmatrix}.$$

Proof. Exercise: show that

$$\vec{Y}_h(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

To find $\vec{Y}_p(t)$,

$$\vec{Y}_p(t) = u_1(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + u_2(t) e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Solve

$$u_1'(t)\vec{Y}_1 + u_2'(t)\vec{Y}_2 = \vec{F}(t).$$

Thus

$$u_1' \begin{bmatrix} 1 \\ 2 \end{bmatrix} + u_2' e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{t} \\ 5 + \frac{2}{t} \end{bmatrix}.$$

So

$$\begin{cases} u_1' - 2u_2'e^{-5t} = \dfrac{1}{t}, & (1) \\ 2u_1' + u_2'e^{-5t} = 5 + \dfrac{2}{t}. & (2) \end{cases}$$

Now compute $2(1) - (2)$:

$$-5u_2'e^{-5t} = \frac{2}{t} - 5 - \frac{2}{t} = -5.$$

Hence

$$u_2' = e^{5t} \quad \Rightarrow \quad u_2(t) = \int e^{5t}\,dt = \frac{1}{5}e^{5t}.$$

Using (1),

$$u_1' - 2(u_2')e^{-5t} = \frac{1}{t} \quad \Rightarrow \quad u_1' - 2(e^{5t})e^{-5t} = \frac{1}{t} \quad \Rightarrow \quad u_1' = \frac{1}{t} + 2.$$

Therefore

$$u_1(t) = \ln|t| + 2t.$$

So

$$\vec{Y}_p(t) = (\ln|t| + 2t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{5}e^{5t}e^{-5t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Thus

$$\vec{Y}_p(t) = (\ln|t| + 2t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The general solution is

$$\vec{Y}(t) = \vec{Y}_h(t) + \vec{Y}_p(t),$$

that is,

$$\vec{Y}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2e^{-5t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + (\ln|t| + 2t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

$\square$

Exercise 5.4. Solve

$$\vec{Y}\,'(t) = \begin{bmatrix} 2 & 1 \\ -5 & 0 \end{bmatrix}\vec{Y}(t) + \begin{bmatrix} 0 \\ 4e^t \end{bmatrix}.$$

(a) using the method of variation of parameters.

(b) using the method of undetermined coefficients

$$\left(\vec{Y}_p(t) = \begin{bmatrix} A_1e^t \\ A_2e^t \end{bmatrix}\right).$$

5.2.3 Solving Syste of Equations using Laplace Transform

Example 5.10. Solve

$$\begin{cases} x_1''(t) + 2x_1(t) - x_2(t) = 0, \\ x_2''(t) - x_1(t) + 2x_2(t) = 0, \end{cases}$$

with ICs:

$$\begin{cases} x_1(0) = 0, \quad x_1'(0) = -1, \\ x_2(0) = 0, \quad x_2'(0) = 1. \end{cases}$$

Solution. Apply LT to the SDE:

$$\begin{cases} \mathcal{L}\{x_1''\} + 2\mathcal{L}\{x_1\} - \mathcal{L}\{x_2\} = 0, \\ \mathcal{L}\{x_2''\} - \mathcal{L}\{x_1\} + 2\mathcal{L}\{x_2\} = 0. \end{cases}$$

Let

$$X_1(s) = \mathcal{L}\{x_1(t)\}, \qquad X_2(s) = \mathcal{L}\{x_2(t)\}.$$

Using

$$\mathcal{L}\{x_i''(t)\} = s^2 X_i(s) - s x_i(0) - x_i'(0),$$

and the initial conditions

$$x_1(0) = 0, \quad x_1'(0) = -1, \qquad x_2(0) = 0, \quad x_2'(0) = 1,$$

we get

$$\begin{cases} s^2 X_1(s) - s x_1(0) - x_1'(0) + 2X_1(s) - X_2(s) = 0, \\ s^2 X_2(s) - s x_2(0) - x_2'(0) - X_1(s) + 2X_2(s) = 0. \end{cases}$$

Hence

$$\begin{cases} (s^2+2)X_1(s) = X_2(s) - 1, \\ (s^2+2)X_2(s) = X_1(s) + 1. \end{cases}$$

Solving this system gives

$$X_1(s) = -\frac{1}{s^2+3}, \qquad X_2(s) = \frac{1}{s^2+3}.$$

Now take inverse Laplace transforms:

$$x_1(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s^2+3}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\},$$

$$x_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\}.$$

Since

$$\mathcal{L}\{\sin(\sqrt{3}\,t)\} = \frac{\sqrt{3}}{s^2+(\sqrt{3})^2} = \frac{\sqrt{3}}{s^2+3},$$

we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\} = \frac{1}{\sqrt{3}}\sin(\sqrt{3}\,t).$$

Therefore

$$\boxed{x_1(t) = -\frac{1}{\sqrt{3}}\sin(\sqrt{3}\,t), \qquad x_2(t) = \frac{1}{\sqrt{3}}\sin(\sqrt{3}\,t).}$$

□

5.2.4 Solving Vector Differential Equations using Laplace Transform

Proposition 5.2. For a vector-valued function with derivative

$$\vec{x}\,'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix},$$

if $\mathcal{L}\{x_1(t)\}$ and $\mathcal{L}\{x_2'(t)\}$ exist, then

$$\mathcal{L}\{\vec{x}\,'(t)\} = s\,\mathcal{L}\{\vec{x}(t)\} - \vec{x}(0).$$

Proposition 5.3. If A is a constant matrix and $\mathcal{L}\{\vec{x}(t)\}$ exists, then

$$\mathcal{L}\{A\vec{x}(t)\} = A\,\mathcal{L}\{\vec{x}(t)\}.$$

Procedure:

$$\vec{x}\,'(t) = A\vec{x}(t), \qquad \vec{x}(0) = \vec{a}.$$

1. Apply LT to both sides of the vector DE:

   $$\mathcal{L}\{\vec{x}\,'(t)\} = \mathcal{L}\{A\vec{x}(t)\} = A\,\mathcal{L}\{\vec{x}(t)\}.$$

   Hence

   $$s\vec{X}(s) - \vec{x}(0) = A\vec{X}(s).$$

   Since $\vec{x}(0) = \vec{a}$, we get

   $$(sI - A)\vec{X}(s) = \vec{a}.$$

2. Assuming $(sI - A)$ is invertible, we can solve for

   $$\vec{X}(s) = (sI - A)^{-1}\vec{a}.$$

3. Take the inverse LT to find $\vec{x}(t)$.

Example 5.11. Use LT to solve the IVP:

$$\vec{x}\,'(t) = \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix} \vec{x}(t), \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution. Applying LT to the vector DE,

$$\mathcal{L}\{\vec{x}\,'(t)\} = A\,\mathcal{L}\{\vec{x}(t)\}.$$

We need to find

$$\vec{X}(s) = (sI - A)^{-1}\vec{x}(0).$$

Let

$$B = (sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} s+4 & -1 \\ 2 & s+1 \end{bmatrix}.$$

If

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Thus

$$B^{-1} = \frac{1}{(s+4)(s+1)+2} \begin{bmatrix} s+1 & 1 \\ -2 & s+4 \end{bmatrix}.$$

Since

$$(s+4)(s+1)+2 = (s+2)(s+3),$$

we get

$$B^{-1} = \begin{bmatrix} \dfrac{s+1}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-2}{(s+2)(s+3)} & \dfrac{s+4}{(s+2)(s+3)} \end{bmatrix}.$$

Hence

$$\vec{X}(s) = \begin{bmatrix} \dfrac{s+1}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-2}{(s+2)(s+3)} & \dfrac{s+4}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{s+1}{(s+2)(s+3)} \\ \dfrac{-2}{(s+2)(s+3)} \end{bmatrix}.$$

Therefore

$$\vec{x}(t) = \mathcal{L}^{-1}\{\vec{X}(s)\} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\dfrac{s+1}{(s+2)(s+3)}\right\} \\ \mathcal{L}^{-1}\left\{\dfrac{-2}{(s+2)(s+3)}\right\} \end{bmatrix}.$$

Partial fraction decomposition:

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+2)(s+3)}\right\} \quad \Rightarrow \quad \frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3},$$

so

$$A = -1, \qquad B = 2.$$

Thus

$$x_1(t) = -\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = -e^{-2t} + 2e^{-3t}.$$

Also,

$$x_2(t) = \mathcal{L}^{-1}\left\{\frac{-2}{(s+2)(s+3)}\right\} \quad \Rightarrow \quad \frac{-2}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3},$$

so

$$A = -2, \qquad B = 2.$$

Hence

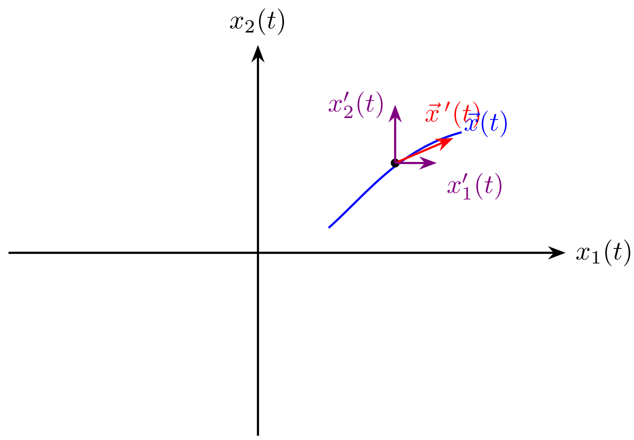
$$x_2(t) = -2\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = -2e^{-2t} + 2e^{-3t}.$$

Therefore

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -e^{-2t} + 2e^{-3t} \\ -2e^{-2t} + 2e^{-3t} \end{bmatrix}.$$

□

5.3 Direction field (Phase Portrait for Vector DEs)



$$\vec{x}\,'(t) = A\vec{x}(t)$$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\vec{x}\,'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

Isoclines:

$$\begin{cases} \text{vertical isocline (VI)} \qquad x_1'(t) = 0, \\ \text{horizontal isocline (HI)} \qquad x_2'(t) = 0. \end{cases}$$

Vector DE:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t), \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t). \end{cases}$$

Lecture 22 - Tuesday, March 24

The goal of this lecture is to understand the *phase portrait* (or direction field) of a 2×2 linear system. Rather than solving only for formulas, we want to understand the geometry of the solutions in the x_1x_2-plane.

Example 5.12. Solve
$$\theta'' + \omega_0^2\theta = 0, \qquad \theta = \theta(t), \qquad \omega_0 = \frac{g}{\ell}$$

Proof. Introduce
$$x_1 = \theta, \qquad x_2 = \theta'.$$
Then
$$x_1' = \theta' = x_2, \qquad x_2' = \theta'' = -\omega_0^2\theta = -\omega_0^2 x_1.$$
So the system becomes
$$\begin{cases} x_1' = x_2, \\ x_2' = -\omega_0^2 x_1. \end{cases}$$
Equivalently,
$$\vec{x}\,'(t) = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \vec{x}(t).$$

Isoclines. The vertical isocline is given by
$$x_1' = 0 \iff x_2 = 0.$$
The horizontal isocline is given by
$$x_2' = 0 \iff x_1 = 0.$$

Direction of motion on the axes.

- If $x_2 = 0$ and $x_1 > 0$, then
$$x_2' = -\omega_0^2 x_1 < 0,$$
so the arrows point downward.
- If $x_2 = 0$ and $x_1 < 0$, then
$$x_2' = -\omega_0^2 x_1 > 0,$$
so the arrows point upward.
- If $x_1 = 0$ and $x_2 > 0$, then
$$x_1' = x_2 > 0,$$
so the arrows point to the right.
- If $x_1 = 0$ and $x_2 < 0$, then
$$x_1' = x_2 < 0,$$
so the arrows point to the left.

Direction in the four quadrants.

- At $(1,1)$, we have $x_1' > 0$ and $x_2' < 0$, so the motion is right and down.
- At $(1,-1)$, we have $x_1' < 0$ and $x_2' < 0$, so the motion is left and down.
- At $(-1,-1)$, we have $x_1' < 0$ and $x_2' > 0$, so the motion is left and up.
- At $(-1,1)$, we have $x_1' > 0$ and $x_2' > 0$, so the motion is right and up.

Hence the trajectories circulate around the origin. Since this is an undamped system, the origin is not attracting or repelling: solutions move on closed curves around the equilibrium. Thus the origin is a *center*. $\square$

Example 5.13. Consider

$$x'' + 2\zeta\omega_0 x' + \omega_0^2 x = 0, \qquad \omega_0 = 1.$$

For the overdamped case in the notes, $\zeta = \frac{5}{4}$, so the equation becomes

$$x'' + \frac{5}{2}x' + x = 0.$$

Solution. Let

$$x_1 = x, \qquad x_2 = x'.$$

Then

$$x_1' = x_2, \qquad x_2' = x'' = -x - \frac{5}{2}x' = -x_1 - \frac{5}{2}x_2.$$

So

$$\begin{cases} x_1' = x_2, \\ x_2' = -x_1 - \frac{5}{2}x_2, \end{cases}$$

or

$$\vec{x}\,'(t) = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix} \vec{x}(t).$$

General solution. The characteristic equation is

$$\lambda^2 + \frac{5}{2}\lambda + 1 = 0.$$

Its roots are

$$\lambda_1 = -\frac{1}{2}, \qquad \lambda_2 = -2.$$

Thus the eigenvalues are real, negative, and distinct. The corresponding eigenvectors may be taken as

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Hence the general solution is

$$\vec{x}(t) = c_1 e^{-t/2} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Exceptional solutions. The special straight-line solutions are obtained by setting one coefficient equal to zero.

$$c_1 = c_2 = 0 \quad \Longrightarrow \quad \vec{x}(t) = \vec{0},$$

which is the equilibrium solution.

If $c_2 = 0$ and $c_1 \neq 0$, then

$$\vec{x}(t) = c_1 e^{-t/2} \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

which lies on the line

$$x_2 = -\frac{1}{2}x_1.$$

If $c_1 = 0$ and $c_2 \neq 0$, then

$$\vec{x}(t) = c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

which lies on the line

$$x_2 = -2x_1.$$

These are called the *exceptional solutions.* They are important because they organize the shape of the phase portrait.

Asymptotic behavior. As $t \to \infty$, both exponentials decay to 0, so

$$\vec{x}(t) \to \vec{0}.$$

Therefore the equilibrium at the origin is *asymptotically stable.*

Also, since $e^{-t/2}$ decays more slowly than e^{-2t}, the term involving $e^{-t/2}$ dominates for large t. Therefore most trajectories approach the origin tangent to the line

$$x_2 = -\frac{1}{2}x_1.$$

This line is the *attracting orbit.*

Isoclines. We compute:

$$x_1' = 0 \iff x_2 = 0,$$

so the vertical isocline is

$$x_2 = 0.$$

Also,

$$x_2' = 0 \iff -x_1 - \frac{5}{2}x_2 = 0 \iff x_2 = -\frac{2}{5}x_1.$$

So the horizontal isocline is

$$x_2 = -\frac{2}{5}x_1.$$

Direction of motion.

- Along $x_2 = 0$:
$$x_2' = -x_1.$$
So if $x_1 > 0$, arrows point downward; if $x_1 < 0$, arrows point upward.

- Along $x_2 = -\frac{2}{5}x_1$:
$$x_1' = x_2 = -\frac{2}{5}x_1.$$
So if $x_1 > 0$, arrows point left; if $x_1 < 0$, arrows point right.

By checking sample points in each region, we obtain the direction field and conclude that all nonzero solutions move toward the origin, forming a stable node. □

Example 5.14. Now consider
$$\vec{x}\,'(t) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \vec{x}(t).$$
In coordinates,
$$\begin{cases} x_1' = -x_1 + x_2, \\ x_2' = -x_1 - x_2. \end{cases}$$

Solution. The matrix has eigenvalues
$$\lambda = -1 \pm i.$$
Thus the real part is negative, so we expect solutions to spiral inward toward the origin.

General solution. A real-valued form of the solution is
$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$
This can also be written in amplitude-phase form as
$$\vec{x}(t) = Re^{-t} \begin{bmatrix} \cos(t-\phi) \\ -\sin(t-\phi) \end{bmatrix},$$
where
$$R = \sqrt{c_1^2 + c_2^2}, \qquad c_1 = R\cos\phi, \qquad c_2 = R\sin\phi.$$

Interpretation. The factor e^{-t} shrinks the magnitude to 0, while the sine and cosine terms produce rotation. Therefore trajectories spiral inward. The origin is again asymptotically stable, but this time it is a *spiral sink* rather than a node.

Exceptional solutions. Unlike the real-eigenvalue case, there are no nontrivial straight-line orbits here. The only equilibrium solution is
$$\vec{x}(t) = \vec{0}.$$

Isoclines. From
$$x_1' = -x_1 + x_2, \qquad x_2' = -x_1 - x_2,$$
we get:
$$x_1' = 0 \iff -x_1 + x_2 = 0 \iff x_2 = x_1,$$
so the vertical isocline is
$$x_2 = x_1.$$

Also,
$$x_2' = 0 \iff -x_1 - x_2 = 0 \iff x_2 = -x_1,$$
so the horizontal isocline is
$$x_2 = -x_1.$$

Direction on the isoclines.

- Along $x_2 = x_1$, we have
$$x_2' = -x_1 - x_2 = -2x_1.$$
Thus if $x_1 > 0$, arrows point downward; if $x_1 < 0$, arrows point upward.
- Along $x_2 = -x_1$, we have
$$x_1' = -x_1 + x_2 = -2x_1.$$
Thus if $x_1 > 0$, arrows point left; if $x_1 < 0$, arrows point right.

Checking sample points in each of the four regions shows the rotational behavior. Because the real part of the eigenvalues is negative, the spirals move inward as $t \to \infty$.

Final picture. We now see the three main geometric behaviors appearing in these examples:

- **Purely imaginary eigenvalues:** closed orbits around the origin (center).
- **Real negative eigenvalues:** trajectories move directly toward the origin (stable node).
- **Complex eigenvalues with negative real part:** trajectories spiral into the origin (spiral sink).

So the phase portrait is determined not only by the differential equation itself, but especially by the eigenvalues and eigenvectors of the coefficient matrix. The isoclines help us determine the direction of motion locally, while the eigenstructure explains the global shape of the trajectories. □

Lecture 23 - Thursday, March 26

Example 5.15. Consider
$$\vec{x}\,'(t) = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \vec{x}(t).$$

Solution. **Step 1: general solution.** From the notes, one finds that

$$\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

So the eigenvalues are

$$\lambda_1 = -2, \qquad \lambda_2 = 4.$$

These are real and distinct, and since one is negative while the other is positive, we expect a saddle point.
Step 2: exceptional solutions. There are three special cases.

If

$$c_1 = c_2 = 0,$$

then

$$\vec{x}(t) = \vec{0},$$

which is the equilibrium solution.

If

$$c_1 \neq 0, \qquad c_2 = 0,$$

then

$$\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so the trajectory lies on the line

$$x_2 = x_1.$$

This is a straight-line solution.

If

$$c_1 = 0, \qquad c_2 \neq 0,$$

then

$$\vec{x}(t) = c_2 e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so the trajectory lies on the line

$$x_2 = -x_1.$$

This is the other straight-line solution.
Step 3: stability. As $t \to \infty$, the term e^{-2t} decays to 0, while the term e^{4t} grows without bound. Therefore, unless $c_2 = 0$, the solution moves away from the origin. So the origin is *unstable.* More precisely, since one eigenvalue is negative and the other is positive, the origin is a

saddle critical point.

The line

$$x_2 = x_1$$

is the stable direction, while the line

$$x_2 = -x_1$$

is the unstable direction.

Step 4: isoclines. Write the system as

$$\begin{cases} x_1' = x_1 - 3x_2, \\ x_2' = -3x_1 + x_2. \end{cases}$$

The vertical isocline is defined by

$$x_1' = 0.$$

Thus

$$x_1 - 3x_2 = 0 \quad \Longrightarrow \quad x_2 = \frac{1}{3}x_1.$$

The horizontal isocline is defined by

$$x_2' = 0.$$

Thus

$$-3x_1 + x_2 = 0 \quad \Longrightarrow \quad x_2 = 3x_1.$$

So the isoclines are

$$\text{VI: } x_2 = \frac{1}{3}x_1, \qquad \text{HI: } x_2 = 3x_1.$$

Step 5: direction in each region. To determine the direction of the arrows, we check the sign of x_1' and x_2'.

Along

$$x_2 = 3x_1,$$

we have $x_2' = 0$, so the arrows are horizontal. For example, at $(1, 3)$,

$$x_1' = 1 - 9 = -8 < 0,$$

so the arrows point left. At $(-1, -3)$,

$$x_1' = -1 + 9 = 8 > 0,$$

so the arrows point right.

Along

$$x_2 = \frac{1}{3}x_1,$$

we have $x_1' = 0$, so the arrows are vertical. For example, at $(1, \frac{1}{3})$,

$$x_2' = -3 + \frac{1}{3} = -\frac{8}{3} < 0,$$

so the arrows point downward. At $(-1, -\frac{1}{3})$,

$$x_2' = 3 - \frac{1}{3} = \frac{8}{3} > 0,$$

so the arrows point upward.

Now choose one test point in each region:

At $(2, 3)$,

$$x_1' = 2 - 9 = -7 < 0, \qquad x_2' = -6 + 3 = -3 < 0,$$

so the arrows point left and down.

At $(2, -3)$,

$$x_1' = 2 + 9 = 11 > 0, \qquad x_2' = -6 - 3 = -9 < 0,$$

so the arrows point right and down.

At $(-2, -3)$,

$$x_1' = -2 + 9 = 7 > 0, \qquad x_2' = 6 - 3 = 3 > 0,$$

so the arrows point right and up.

At $(-2, 3)$,

$$x_1' = -2 - 9 = -11 < 0, \qquad x_2' = 6 + 3 = 9 > 0,$$

so the arrows point left and up.

Putting this together gives the saddle-shaped phase portrait shown in the notes. The trajectories are attracted toward the origin along

$$x_2 = x_1$$

and repelled away from the origin along

$$x_2 = -x_1.$$

□

Example 5.16. Now consider

$$\vec{x}\,'(t) = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix} \vec{x}(t).$$

Solution. **Step 1: general solution.** From the notes, the solution is

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right).$$

So there is a repeated eigenvalue

$$\lambda_1 = \lambda_2 = -1.$$

Step 2: exceptional solutions. If

$$c_1 = c_2 = 0,$$

then

$$\vec{x}(t) = \vec{0},$$

which is again the equilibrium point.

If

$$c_2 = 0, \qquad c_1 \neq 0,$$

then

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

so the trajectory lies on the line

$$x_2 = \frac{1}{2} x_1.$$

This is the only straight-line exceptional solution.

If

$$c_1 = 0, \qquad c_2 \neq 0,$$

then

$$\vec{x}(t) = c_2 e^{-t} \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right),$$

which is not a straight line, so it does not yield another exceptional solution. This is typical of the defective repeated-eigenvalue case.

Step 3: stability. As $t \to \infty$, every term contains the factor e^{-t}, so

$$\vec{x}(t) \to \vec{0}.$$

Hence the equilibrium point is

asymptotically stable.

Moreover, for large t, the term involving

$$t e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

dominates the correction term, so trajectories approach the origin tangent to the line

$$x_2 = \frac{1}{2} x_1.$$

Thus

$$x_2 = \frac{1}{2} x_1$$

is the attracting orbit.

Step 4: isoclines. Write the system as

$$\begin{cases} x_1' = -3x_1 + 4x_2, \\ x_2' = -x_1 + x_2. \end{cases}$$

The vertical isocline is obtained from

$$x_1' = 0: \qquad -3x_1 + 4x_2 = 0 \quad \Longrightarrow \quad x_2 = \frac{3}{4} x_1.$$

The horizontal isocline is obtained from

$$x_2' = 0: \qquad -x_1 + x_2 = 0 \quad \Longrightarrow \quad x_2 = x_1.$$

So the isoclines are

$$\text{VI: } x_2 = \frac{3}{4}x_1, \qquad \text{HI: } x_2 = x_1.$$

Step 5: direction in each region. Along

$$x_2 = \frac{3}{4}x_1,$$

we have $x_1' = 0$, so arrows are vertical. At $(1, \frac{3}{4})$,

$$x_2' = -1 + \frac{3}{4} = -\frac{1}{4} < 0,$$

so the arrows point downward. At $(-1, -\frac{3}{4})$,

$$x_2' = 1 - \frac{3}{4} = \frac{1}{4} > 0,$$

so the arrows point upward.

Along

$$x_2 = x_1,$$

we have $x_2' = 0$, so arrows are horizontal. At $(1, 1)$,

$$x_1' = -3 + 4 = 1 > 0,$$

so the arrows point right. At $(-1, -1)$,

$$x_1' = 3 - 4 = -1 < 0,$$

so the arrows point left.

Now choose one point in each region:

At $(1, 0)$,

$$x_1' = -3 < 0, \qquad x_2' = -1 < 0,$$

so the arrows point down and left.

At $(-1, 0)$,

$$x_1' = 3 > 0, \qquad x_2' = 1 > 0,$$

so the arrows point up and right.

At $(1, 2)$,

$$x_1' = -3 + 8 = 5 > 0, \qquad x_2' = -1 + 2 = 1 > 0,$$

so the arrows point up and right.

At $(1, \frac{1}{2})$,

$$x_1' = -3 + 2 = -1 < 0, \qquad x_2' = -1 + \tfrac{1}{2} = -\tfrac{1}{2} < 0,$$

so the arrows point down and left.

This produces the node-type phase portrait shown in the notes. Since the repeated eigenvalue is negative, all trajectories approach the origin, and they do so tangent to the attracting orbit

$$x_2 = \frac{1}{2}x_1.$$

□

5.3.1 Classification summary for phase portraits

The notes also include a useful classification table for the system

$$\vec{x}\,' = A\vec{x}.$$

The basic cases are:

Eigenvalues	Type of critical point	Stability
$\lambda_1 > \lambda_2 > 0$	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or improper node	Asymptotically stable
$\lambda_{1,2} = a \pm ib,\ a > 0$	Spiral point	Unstable
$\lambda_{1,2} = a \pm ib,\ a < 0$	Spiral point	Asymptotically stable
$\lambda_{1,2} = \pm ib$	Centre	Stable

This chart is a very helpful summary: the sign of the real part tells us whether solutions move toward or away from the equilibrium, and the presence of an imaginary part determines whether the motion spirals or not.

6 Series Solutions for ODEs

We now begin the method of power series solutions. Consider a second-order differential equation of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

where $p(x)$ and $q(x)$ are polynomial functions. In such a situation, it is natural to try a power series solution.

Example 3: solve $y''(x) + y(x) = 0$ using a series solution. We assume that the solution has the form

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n,$$

where the series converges for

$$|x| < R$$

for some radius of convergence R (possibly $R = \infty$). Differentiate term-by-term:

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute into the differential equation

$$y''(x) + y(x) = 0.$$

This gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

To combine these into a single series, shift the index in the first sum by setting

$$m = n - 2, \qquad \text{so that} \qquad n = m + 2.$$

Then

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m.$$

Renaming the index m back to n, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Hence

$$\sum_{n=0}^{\infty} \Big[(n+2)(n+1) a_{n+2} + a_n\Big] x^n = 0.$$

Since this power series is identically zero, each coefficient must vanish:

$$(n+2)(n+1)a_{n+2} + a_n = 0.$$

Therefore the recurrence relation is

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \qquad n = 0, 1, 2, \ldots.$$

Even and odd coefficients. This recurrence links coefficients two indices apart, so the even coefficients depend only on a_0, and the odd coefficients depend only on a_1.

For the even coefficients:

$$a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{a_0}{2},$$
$$a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!},$$

and in general

$$a_{2n} = (-1)^n \frac{a_0}{(2n)!}.$$

For the odd coefficients:

$$a_3 = -\frac{a_1}{3 \cdot 2} = -\frac{a_1}{3!},$$
$$a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5!},$$

and in general

$$a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}.$$

So the full series becomes

$$\begin{aligned} y(x) = a_0 &\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \\ &+ a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right). \end{aligned}$$

Recognizing these familiar Taylor series, we conclude that

$$y(x) = a_0 \cos x + a_1 \sin x.$$

Comment 6.1. This agrees with the solution we already know from elementary methods, which is a good check that the power series method is working correctly.

Lecture 24 - Tuesday, March 31

Example 6.1. Solve the DE using series solutions:

$$y'' - 2xy' + y(x) = 0.$$

Solution. Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Then

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

Substituting into the DE gives

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - 2x \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Re-indexing,

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n - 2 \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Hence

$$\sum_{n=0}^{\infty} \Big[a_{n+2}(n+2)(n+1) + a_n(1-2n) \Big] x^n = 0.$$

Therefore,

$$a_{n+2}(n+2)(n+1) + a_n(1-2n) = 0,$$

so

$$a_{n+2} = \frac{2n-1}{(n+2)(n+1)}\, a_n, \qquad n = 0, 1, 2, \ldots$$

Thus the even and odd coefficients separate:

$$a_0 \to a_2 \to a_4 \to \cdots , \qquad a_1 \to a_3 \to a_5 \to \cdots$$

For the even coefficients,

$$n = 0 \implies a_2 = -\frac{1}{2} a_0,$$

$$n = 2 \implies a_4 = \frac{3}{(4)(3)} a_2 = -\frac{3}{4!} a_0,$$

$$n = 4 \implies a_6 = \frac{7}{(6)(5)} a_4 = -\frac{1 \cdot 3 \cdot 7}{6!} a_0,$$

and so on. For the odd coefficients,

$$n = 1 \implies a_3 = \frac{1}{(3)(2)} a_1 = \frac{1}{3!} a_1,$$

$$n = 3 \implies a_5 = \frac{5}{(5)(4)} a_3 = \frac{1 \cdot 5}{5!} a_1,$$

$$n = 5 \implies a_7 = \frac{9}{(7)(6)} a_5 = \frac{1 \cdot 5 \cdot 9}{7!} a_1,$$

and so on. Substituting back into the series,

$$Y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

becomes

$$Y(x) = a_0 \left[1 - \frac{1}{2}x^2 - \frac{3}{4!}x^4 - \frac{1\cdot 3\cdot 7}{6!}x^6 - \frac{1\cdot 3\cdot 7\cdot 11}{8!}x^8 - \cdots\right]$$
$$+a_1\left[x + \frac{1}{3!}x^3 + \frac{1\cdot 5}{5!}x^5 + \frac{1\cdot 5\cdot 9}{7!}x^7 + \cdots\right].$$

Hence

$$Y(x) = a_0 Y_1(x) + a_1 Y_2(x),$$

where

$$Y_1(x) = 1 - \frac{1}{2}x^2 - \frac{3}{4!}x^4 - \frac{1\cdot 3\cdot 7}{6!}x^6 - \frac{1\cdot 3\cdot 7\cdot 11}{8!}x^8 - \cdots$$

and

$$Y_2(x) = x + \frac{1}{3!}x^3 + \frac{1\cdot 5}{5!}x^5 + \frac{1\cdot 5\cdot 9}{7!}x^7 + \cdots.$$

Also,

$$Y_1(-x) = Y_1(x), \qquad Y_2(-x) = -Y_2(x),$$

so Y_1 is even and Y_2 is odd. □

Example 6.2. Find the solution to Airy's equation using series solutions:

$$y'' - xy(x) = 0.$$

Solution. Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}.$$

Substituting into the DE gives

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Re-indexing,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

Separating the $n = 0$ term in the first series,

$$2a_2 + \sum_{n=1}^{\infty}\Big((n+2)(n+1)a_{n+2} - a_{n-1}\Big)x^n = 0.$$

Hence

$$2a_2 = 0 \implies a_2 = 0,$$

and for $n = 1, 2, 3, \ldots,$

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0,$$

so

$$a_{n+2} = \frac{1}{(n+2)(n+1)}\, a_{n-1}, \qquad n = 1, 2, 3, \ldots$$

Thus the coefficients split into three chains:

$$a_0 \to a_3 \to a_6 \to a_9 \to \cdots, \qquad a_1 \to a_4 \to a_7 \to a_{10} \to \cdots, \qquad a_2 \to a_5 \to a_8 \to a_{11} \to \cdots$$

and since $a_2 = 0$, the third chain is identically zero. Now,

$$n = 1 \implies a_3 = \frac{1}{(3)(2)} a_0 = \frac{1}{3!} a_0,$$

$$n = 2 \implies a_4 = \frac{1}{(4)(3)} a_1,$$

$$n = 3 \implies a_5 = \frac{1}{(5)(4)} a_2 = 0,$$

$$n = 4 \implies a_6 = \frac{1}{(6)(5)} a_3 = \frac{1}{(6)(5)(3)(2)} a_0,$$

$$n = 5 \implies a_7 = \frac{1}{(7)(6)} a_4 = \frac{1}{(7)(6)(4)(3)} a_1,$$

$$n = 6 \implies a_8 = \frac{1}{(8)(7)} a_5 = 0,$$

$$n = 7 \implies a_9 = \frac{1}{(9)(8)} a_6 = \frac{1}{(9)(8)(6)(5)(3)(2)} a_0,$$

and so on. Therefore

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

becomes

$$y(x) = a_0 Y_1(x) + a_1 Y_2(x),$$

where

$$Y_1(x) = 1 + \frac{1}{(3)(2)} x^3 + \frac{1}{(6)(5)(3)(2)} x^6 + \frac{1}{(9)(8)(6)(5)(3)(2)} x^9 + \cdots$$

and

$$Y_2(x) = x + \frac{1}{(4)(3)} x^4 + \frac{1}{(7)(6)(4)(3)} x^7 + \frac{1}{(10)(9)(7)(6)(4)(3)} x^{10} + \cdots.$$

Hence the general series solution is

$$\boxed{y(x) = c_1 Y_1(x) + c_2 Y_2(x).}$$

□

6.1 Ordinary and Singular Points for 2nd Order ODEs

Consider the second-order linear differential equation

$$a_2(x) y''(x) + a_1(x) y'(x) + a_0(x) y(x) = 0.$$

Suppose $a_2(x)$, $a_1(x)$, and $a_0(x)$ are polynomials in x. Then they are continuous at every $x_0 \in \mathbb{R}$.

Definition 6.1. **[Ordinary Point]**

The point x_0 such that

$$a_2(x_0) \neq 0$$

is called an **ordinary point**.

Definition 6.2. **[Singular Point]**

If instead

$$a_2(x_0) = 0,$$

then x_0 is called a **singular point**.

Dividing the equation by $a_2(x)$, we obtain

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0.$$

Note 6.1. If $x_0 \in \mathbb{R}$ is an ordinary point of the DE, then the series solution method can be used. Hence there exist two linearly independent series solutions $Y_1(x)$ and $Y_2(x)$.
Otherwise, we use the Frobenius method to solve the DE.

6.1.1 Frobenius Method

Consider the second-order linear DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

If $a_2(x) \neq 0$, we divide by $a_2(x)$ and write it in standard form:

$$y'' + p(x)y' + q(x)y = 0, \qquad p(x) = \frac{a_1(x)}{a_2(x)}, \qquad q(x) = \frac{a_0(x)}{a_2(x)}.$$

Suppose x_0 is a singular point. Then:

Definition 6.3. **[Regular Singular]**

x_0 is a **regular singular point** if $\lim_{x\to x_0}(x - x_0)p(x)$ and $\lim_{x\to x_0}(x - x_0)^2 q(x)$ both exist and are finite. Otherwise, x_0 is called an **irregular singular point**.

Comment 6.2. The idea is that at a regular singular point, the singularities in $p(x)$ and $q(x)$ are not too severe: $p(x)$ is allowed to behave like $\frac{1}{x-x_0}$, and $q(x)$ is allowed to behave like $\frac{1}{(x-x_0)^2}$, but not worse.

Example 6.3.

$$xy'' + y = 0.$$

Here

$$a_2(x) = x, \qquad a_1(x) = 0, \qquad a_0(x) = 1.$$

Since

$$a_2(0) = 0,$$

the point $x_0 = 0$ is a singular point.
To classify it, divide by x:

$$y'' + \frac{1}{x} y = 0.$$

So

$$p(x) = 0, \qquad q(x) = \frac{1}{x}.$$

Then

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} 0 = 0, \qquad \lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x = 0.$$

Both limits exist and are finite, so $x_0 = 0$ is a *regular singular point.*

Hence this is a case where the Frobenius method can be used.

Example 6.4.

$$y'' + xy' + 2y = 0.$$

Here

$$a_2(x) = 1, \qquad a_1(x) = x, \qquad a_0(x) = 2.$$

Since $a_2(x) = 1 \neq 0$ for every $x \in \mathbb{R}$, every point $x_0 \in \mathbb{R}$ is an ordinary point.

Therefore, the usual power series method can be used.

Example 6.5.

$$x^2y'' + y' + y = 0.$$

Here

$$a_2(x) = x^2, \qquad a_1(x) = 1, \qquad a_0(x) = 1.$$

Since

$$a_2(0) = 0,$$

the point $x_0 = 0$ is a singular point.
Write the equation in standard form:

$$y'' + \frac{1}{x^2}y' + \frac{1}{x^2}y = 0.$$

Thus

$$p(x) = \frac{1}{x^2}, \qquad q(x) = \frac{1}{x^2}.$$

Now

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} \frac{1}{x},$$

which does not exist as a finite limit, while

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} 1 = 1.$$

Since one of the required limits fails to exist finitely, $x_0 = 0$ is an *irregular singular point*.

The Frobenius method.

When $x_0 = 0$ is a regular singular point, we do not assume an ordinary power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Instead, we try the more general form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r},$$

where r is a constant to be determined.

Discovery 6.1. At a regular singular point, the true solution may start with a non-integer power x^r, not necessarily with x^0. The Frobenius method allows for this extra factor x^r, so it is more flexible than the ordinary power series method.

Note 6.2. The Frobenius method is designed for *regular singular points*, not arbitrary singular points. If the point is *irregular singular*, the Frobenius method does not generally apply directly.

Example 6.6. Solving the following DE using the method of Frobenius:

$$4xy'' + 2y' + y(x) = 0$$

Solution. Consider the differential equation

$$4x\,y'' + 2y' + y = 0.$$

We begin by identifying

$$a_2(x) = 4x, \qquad a_1(x) = 2, \qquad a_0(x) = 1.$$

Since

$$a_2(0) = 0,$$

the point $x = 0$ is a singular point. We now check whether it is a *regular* singular point.

First, rewrite the equation in standard form:

$$y'' + \frac{2}{4x}y' + \frac{1}{4x}y = 0,$$

so that

$$p(x) = \frac{1}{2x}, \qquad q(x) = \frac{1}{4x}.$$

Then

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x \cdot \frac{1}{2x} = \frac{1}{2},$$

and

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \cdot \frac{1}{4x} = \lim_{x\to 0} \frac{x}{4} = 0.$$

Both limits exist, so $x = 0$ is a regular singular point. Therefore, the Frobenius method applies.

We now look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Differentiating term-by-term gives

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

and

$$y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$$

Substituting these into the differential equation, we obtain

$$4x\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2} + 2\sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

After simplifying the powers of x, this becomes

$$\sum_{n=0}^{\infty} \big[4a_n(n+r)(n+r-1) + 2a_n(n+r)\big] x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

To combine the two series, shift the index in the first sum by writing $n \mapsto n+1$. This gives

$$\begin{aligned}&\big[4a_0 r(r-1) + 2a_0 r\big] x^{r-1} \\ &\quad + \sum_{n=0}^{\infty} \big[4a_{n+1}(n+r+1)(n+r) + 2a_{n+1}(n+r+1) + a_n\big] x^{n+r} = 0.\end{aligned}$$

Since this series must vanish for all x, each coefficient must be zero. From the lowest-power term x^{r-1}, we get the indicial equation:

$$4a_0 r(r-1) + 2a_0 r = 0.$$

Assuming $a_0 \neq 0$, this simplifies to

$$2r(2r-1) = 0.$$

Hence the roots are

$$r = 0, \qquad r = \frac{1}{2}.$$

For the remaining coefficients, we obtain the recurrence relation

$$4a_{n+1}(n+r+1)(n+r) + 2a_{n+1}(n+r+1) + a_n = 0,$$

or equivalently,

$$a_{n+1} = -\frac{a_n}{2(n+r+1)(2n+2r+1)}.$$

First solution: $r = 0$. When $r = 0$, the recurrence becomes

$$a_{n+1} = -\frac{a_n}{2(n+1)(2n+1)}.$$

The first few coefficients are

$$a_1 = -\frac{a_0}{2!}, \qquad a_2 = \frac{a_0}{4!}, \qquad a_3 = -\frac{a_0}{6!},$$

and in general,

$$a_n = \frac{(-1)^n a_0}{(2n)!}.$$

Therefore,

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}. \end{aligned}$$

Recognizing this as the power series for $\cos(\sqrt{x})$, we get

$$y_1(x) = a_0 \cos(\sqrt{x}).$$

Second solution: $r = \frac{1}{2}$**.** When $r = \frac{1}{2}$, the recurrence becomes

$$a_{n+1} = -\frac{a_n}{2\left(n+\frac{3}{2}\right)(2n+2)} = -\frac{a_n}{(2n+3)(2n+2)}.$$

The first few coefficients are

$$a_1 = -\frac{a_0}{3!}, \qquad a_2 = \frac{a_0}{5!}, \qquad a_3 = -\frac{a_0}{7!},$$

and hence

$$a_n = \frac{(-1)^n a_0}{(2n+1)!}.$$

Thus,

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+\frac{1}{2}}}{(2n+1)!} \\ &= a_0 \sin(\sqrt{x}). \end{aligned}$$

Therefore, the general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

that is,

$$\boxed{y(x) = A_1 \cos(\sqrt{x}) + A_2 \sin(\sqrt{x})}.$$

□

Example 6.7. Determine the singular point of the following DE and solve it using frobenius method:

$$x^2 y'' + xy' + (x^2 - 1)y(x) = 0$$

Solution. Consider the differential equation

$$x^2 y'' + xy' + (x^2 - 1)y = 0.$$

We first identify

$$a_2(x) = x^2, \qquad a_1(x) = x, \qquad a_0(x) = x^2 - 1.$$

Since

$$a_2(0) = 0,$$

it follows that $x = 0$ is a singular point. We now check whether it is a *regular* singular point.

To do this, rewrite the equation in standard form:

$$y'' + \frac{x}{x^2}y' + \frac{x^2-1}{x^2}y = 0,$$

so that

$$p(x) = \frac{1}{x}, \qquad q(x) = \frac{x^2-1}{x^2}.$$

Now compute

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x \cdot \frac{1}{x} = 1,$$

and

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \cdot \frac{x^2-1}{x^2} = \lim_{x\to 0}(x^2-1) = -1.$$

Both limits exist, so $x = 0$ is a regular singular point. Therefore, the Frobenius method applies.

We now assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Then

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

and

$$y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$$

Substituting these into the differential equation gives

$$x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + (x^2-1) \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

After simplifying, we obtain

$$\sum_{n=0}^{\infty} a_n (n+r)\big((n+r-1)+1\big) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0,$$

that is,

$$\sum_{n=0}^{\infty} \big[a_n (n+r)^2 - a_n\big] x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0.$$

To combine the powers, reindex the second sum by replacing n with $n-2$:

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+r}.$$

Thus,

$$\sum_{n=0}^{\infty}\big[a_n(n+r)^2 - a_n\big]x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} = 0.$$

Now separate the first two terms from the first series:

$$\big(a_0r^2 - a_0\big)x^r + \big(a_1(1+r)^2 - a_1\big)x^{r+1} + \sum_{n=2}^{\infty}\big[a_n(n+r)^2 - a_n + a_{n-2}\big]x^{n+r} = 0.$$

Since this series must vanish for all x, each coefficient must be zero.

First, from the coefficient of x^r, we get

$$a_0(r^2 - 1) = 0.$$

Assuming $a_0 \neq 0$, this gives the indicial equation

$$r^2 - 1 = 0,$$

so the roots are

$$r_1 = 1, \qquad r_2 = -1.$$

Next, from the coefficient of x^{r+1}, we obtain

$$a_1\big((1+r)^2 - 1\big) = 0.$$

Checking both roots $r = \pm 1$, we see that in either case this forces

$$a_1 = 0.$$

Finally, for $n \geq 2$, we obtain the recurrence relation

$$a_n(n+r)^2 - a_n + a_{n-2} = 0,$$

or equivalently,

$$a_n = -\frac{a_{n-2}}{(n+r)^2 - 1}.$$

Case 1: $r = 1$. When $r = 1$, the recurrence becomes

$$a_n = -\frac{a_{n-2}}{(n+1)^2 - 1}.$$

In particular,

$$a_2 = -\frac{a_0}{(2+1)^2 - 1} = -\frac{a_0}{8},$$

and

$$a_3 = -\frac{a_1}{(3+1)^2 - 1} = 0.$$

Since $a_1 = 0$, it follows recursively that

$$a_3 = a_5 = a_7 = \cdots = 0.$$

Continuing with the even-indexed coefficients,

$$a_4 = -\frac{a_2}{(4+1)^2 - 1} = \frac{a_0}{8 \cdot 24} = \frac{a_0}{192}.$$

Therefore, the first Frobenius solution is

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \cdots \\ &= a_0 x - \frac{a_0}{8} x^3 + \frac{a_0}{192} x^5 + \cdots . \end{aligned}$$

So, presenting the first three nonzero terms,

$$y_1(x) = a_0 \left(x - \frac{x^3}{8} + \frac{x^5}{192} + \cdots \right).$$

Case 2: $r = -1$. When $r = -1$, the recurrence becomes

$$a_n = -\frac{a_{n-2}}{(n-1)^2 - 1}, \qquad n = 2, 3, \ldots$$

Now look at the case $n = 2$. The recurrence relation in its original form gives

$$a_2(2-1)^2 - a_2 + a_0 = 0.$$

That is,

$$a_2(1) - a_2 + a_0 = 0,$$

so

$$a_0 = 0.$$

But this contradicts our assumption $a_0 \neq 0$. Hence for $r = -1$, we do not obtain a second independent Frobenius solution.

Therefore, the second solution must be found by another method, such as reduction of order. If we write

$$y(x) = u(x)\, y_1(x),$$

and substitute into the differential equation, then solving for $u(x)$ produces a second linearly independent solution $y_2(x)$. Accordingly, the general solution can be written as

$$y(x) = A_1 y_1(x) + A_2 y_2(x),$$

where

$$y_1(x) = a_0 \left(x - \frac{x^3}{8} + \frac{x^5}{192} + \cdots \right),$$

and $y_2(x)$ is obtained by reduction of order. $\square$

A Appendix – Some Calculus Results

A.1 Integration by Parts

Theorem A.1. Integration by Parts

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

Example A.1. We compute $\int x \sin x \, \mathrm{d}x$:

$$\begin{aligned}\int x \sin x \, \mathrm{d}x &= -\int x \, \mathrm{d} \cos x \\ &= -\left[x \cos x - \int \cos x \, \mathrm{d}x\right] \\ &= -x \cos x + \sin x + C.\end{aligned}$$

Example A.2. We compute $\int x \sin x \, \mathrm{d}x$:

$$\begin{aligned}\int e^x \sin(x) \, \mathrm{d}x &= \int \sin(x) \, \mathrm{d}(e^x) \\ &= e^x \sin(x) - \int e^x \, \mathrm{d}(\sin(x)) \\ &= e^x \sin(x) - \int e^x \cos(x) \, \mathrm{d}x \\ &= e^x \sin(x) - \int \cos(x) \, \mathrm{d}(e^x) \\ &= e^x \sin(x) - e^x \cos(x) - \int e^x \, \mathrm{d}(\cos(x)) \\ &= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x) \, \mathrm{d}x.\end{aligned}$$

Therefore

$$\int e^x \sin(x) \, dx = \frac{e^x}{2}\big(\sin(x) - \cos(x)\big) + C.$$

B Laplace Table

$f(t)$	$\mathcal{L}\{f(t)\}$
$H(t)\,C$	$\frac{C}{s}$
$H(t)$	$\frac{1}{s}$
$H(t)e^{ct}$	$\frac{1}{s-c}$
$H(t)t^n$	$\frac{n!}{s^{n+1}}$
$H(t)\cos(bt)$	$\frac{s}{s^2+b^2}$
$H(t)\sin(bt)$	$\frac{b}{s^2+b^2}$
$e^{ct}f(t)$	$F(s-c)$
$H(t-c)f(t-c)$	$e^{-sc}F(s)$
$H(t-c)$	$\frac{1}{s}e^{-sc}$
$H(t-c)f(t)$	$e^{-sc}\,\mathcal{L}\{f(t+c)\}$
$t^n f(t)$	$(-1)^n \frac{d^n F}{ds^n}$
$f'(t)$	$s\,\mathcal{L}\{f(t)\} - f(0)$
$f''(t)$	$s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

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